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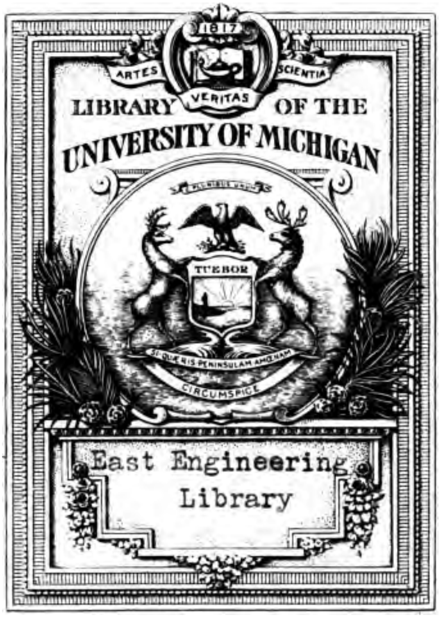

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DIFFERENTIAL AND INTEGRAL CALCULUS.

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DIFFERENTIAL AND INTEGRAL CALCULUS,

WITH APPLICATIONS.

BY

ALFRED GEORGE GREENHILL,
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PROFESSOR OF MATHEMATICS TO THE SENIOR CLASS OF ARTILLERY OFFICERS, WOOLWICH.

SECOND EDITION.

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PREFACE.

The present Treatise is intended as an introduction to the study of the Differential and Integral Calculus, but will be found to contain what it is necessary to know in order to pass on to the subjects which presume a knowledge of the Calculus.

I have endeavoured to make this book suitable not only for the mathematical student, but also for men like engineers and electricians who require the subject for practical applications, to whom even a slight knowledge of the notation and methods of the Calculus is becoming more and more indispensable.

Hitherto in this country the influence of Newton, although the inventor of Fluxions, has been employed to delay the study of this subject and make a knowledge of it the privilege of a select few; my object in writing this treatise has been mainly to present the subject in as simple a manner as possible, in order to encourage a larger number of students to cultivate it.

With the object of keeping the size of the book within reasonable limits, it is assumed that the reader has already acquired a knowledge of the elements of Algebra, Trigonometry, and Coordinate Geometry, as given, for instance, in the treatises of Hall and Knight, J. B. Lock and C. Smith; accordingly I have at once proceeded to the operation and application of Differentiation with as little preliminary explanation as possible.

I have followed the recent American treatises on this subject of Rice and Woolsey Johnson, Byerly and J. H. Taylor, in introducing the notion of Time as an independent variable, and the associated ideas of velocity and acceleration, in order to afford illustrations of the use of the Calculus; this is after all only a return to the Method of Fluxions as invented by Newton, and carried out by Maclaurin and other writers in this country, until supplanted by the notation of the Differential Coefficients of the foreign mathematicians.

The Doctrine of Fluxions is a useful and rigorous method of presenting the elementary ideas of the flow of varying quantities, and is employed in the treatises of Rice and Woolsey Johnson under the name of the Method of Rates; but the notation for a fluxion, for instance $\dot{x}$ the fluxion of x, though easily written is difficult to print, and has the inconvenience of not indicating the independent variable, so that the notation of Leibnitz, $\frac{dx}{dt}$ instead of x, is now used almost universally in printed books; and to economise space, this notation it is now proposed to print in the form dx/dt.

The chief novelties in the present work consist, first, in carrying on the subjects of the Differential and of the Integral Calculus together, instead of, as is usual, completing the Differential before passing on to the Integral Calculus; secondly, in the use of the hyperbolic functions in conjunction with the ordinary circular trigonometrical functions, as thereby an exact analogy is preserved, which is not apparent when only the exponential and logarithmic functions are employed.

The notation of sinh, cosh, tanh, etc., to denote the hyperbolic sine, cosine, tangent, etc., has been employed, in accordance with what appears to be now the most universal custom.

I have ventured also, on the grounds of symmetry, to introduce the inverse hyperbolic functions, and, following Ferrers and Byerly, to denote them by $\sinh^{-1}$, $\cosh^{-1}$, $\tanh^{-1}$, etc., by analogy with $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, etc.; this idea of symmetrical symbolism will be found indicated in Bertrand's Integral Calculus, Chapter I., but has not been pursued apparently because of the lengthiness of the notation there employed, namely, sect. sin hyp., sect. cos hyp., sect. tang hyp., etc., instead of the above.

By the use of the direct and inverse hyperbolic functions in conjunction with the direct and inverse circular functions, the Calculus is in my opinion considerably simplified; and the student is led on more naturally and readily to the consideration of the elliptic and other functions. The consideration of these last functions is however beyond the scope of the present treatise.

To exhibit more clearly the analogy and symmetry between the circular and hyperbolic functions, I have made a digression in Chapter I. on the formulas of the addition equation (as it may be called by analogy with elliptic functions) of ordinary trigonometry, showing how the formulas may all be deduced from a single figure, with the corresponding relations of the hyperbolic functions.

Numerous collections of examples will be found throughout the book, introduced at each point to illustrate what has immediately gone before, and chosen as having some bearing on subsequent stages of mathematics.

The order of arrangement will be found in some respects different to what is customary; for instance, the idea of tracing simple curves from their equations has been introduced into the first chapter, so far as is required for the ordinary applications of the Integral Calculus to finding the areas, etc., of these curves; the general theory of curves being resumed in Chapter VI. Maximums and Minimums also have been investigated without the aid of Taylor's Theorem.

Change of the Independent Variable has only been touched upon where necessary; the general theory of Change of the Independent Variable, Lagrange's and Laplace's Theorems, and the Elimination of Constants and Functions, have been omitted as beyond the scope of an elementary practical treatise.

I have to thank Mr. A. G. Hadcock, Inspector of Ordnance Machinery, Royal Artillery, for drawing the diagrams, and also for revising the proof sheets and preparing the index.

WOOLWICH,
December, 1885.

PREFACE TO THE SECOND EDITION.

In this second edition a rearrangement has been made, by which the general theory of Integration has been relegated to a supplementary final Chapter; only a slight preliminary sketch of Integration is now given in Chapter II., sufficient to carry on the student to some simple applications, and to enable him to follow the subsequent articles.

In this rearrangement, and in many other details at the outset, I have received valuable advice and assistance from Mr. W. Gallatly, M.A.

The theories of Change of the Variable, the Elimination of Constants and Functions, and the Theorems of Lagrange and Laplace have now been included, so that this new edition is increased in size.

By following the custom of giving the examples in smaller print, this increase in size might have been apparently much reduced; but it was decided to use a uniform type throughout the book, as less trying to the eyesight.

The *solidus* notation, for instance dx/dt for $\frac{dx}{dt}$, has now been used sparingly, in places where no confusion or ambiguity could arise, and thereby some economy of space has been obtained.

A few pages of explanation of the simplest Differential Equations which occur in Dynamics and Electricity have been added; while the illustrations of Change of the Variable have been chosen from among the most typical differential equations; and it is hoped that this slight sketch will enable the student to solve an ordinary differential equation which may come in his way.

The greater part of the diagrams have been redrawn by Mr. A. G. Hadcock, who has again helped me in the revision of the proof sheets and the preparation of the Index.

WOOLWICH,
March, 1891.

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Articles in the earlier chapters marked with a * may be omitted at a first reading.

ERRATA.

Page 82, ex. 39, *read*—" with $X = ax^4 + 4bx^3 + 6cx^2 + 4b'x + a'$."
Page 125, line 7, *read*—" $\frac{4}{3}\pi y^3 = \frac{1}{6}\pi \times PP'^3$; ".
Page 139, ex. 8, *read*—" $\dfrac{d^ny}{dx^n} = \dfrac{(n-1)!\{(1+x)^n - (-1+x)^n\}}{(1-x^2)^n}$."
Page 142, last line but one to be cancelled.
Page 143, ex. 2. iv., *read*—" $y^{(2n+1)} = (-1)^n 2n!\, a^{-2n-1}$."
Page 151, ex. 18, the result as given is incorrect.
Page 160, line 3 from bottom, *read*—" $\frac{1}{2}(V+u)$."
Page 183, line 15, *read*—" $Cu = u_1\int Su_2dx - u_2\int Su_1dx$."
Page 231, line 4 from bottom, *read*—" partial fractions."

DIFFERENTIAL AND INTEGRAL CALCULUS.

CHAPTER I.

DIFFERENTIATION.

1. *Introduction. Constant and Variable Quantities.* The Calculus to be developed in the present treatise is the method of reasoning applicable to variable quantities in a state of continuous change.

We call quantities *variable* when they change gradually by increase or decrease; on the contrary *constant*, when they remain unchanged while others change.

Thus the abscissa and ordinate of points on a parabola vary, while the parameter remains constant.

Again, the distance of a railway station from the terminus, or the latitude and longitude of a rock, of a lightship, or of an observatory, are constant quantities.

But the distance of a railway train from the terminus, or the latitude and longitude of a steamer or traveller are variable quantities.

Constant quantities are generally represented algebraically by the first letters of the alphabet, such as a, b, c, ..., or A, B, C, ..., or α, β, γ, ...; while variable quantities are represented by the last letters, ... t, u, v, w, x, y, z, or ... X, Y, Z, or ... ξ, η, ζ.

2. *Definition of a Function; and of Independent and Dependent Variables.*

One variable quantity, denoted by y or $\mathrm{f}(x)$, is said to be a *function* of another variable quantity, denoted by x, when the value of y or $\mathrm{f}(x)$ depends on the value of x.

The variable x, called the *independent variable* or *argument*, is one to which any value may be arbitrarily assigned.

The variable y or $\mathrm{f}(x)$, called the *dependent variable*, is one of which the value is determined, when the arbitrary value of the independent variable x has been assigned.

We may use the notation $\mathrm{f}x$ instead of $\mathrm{f}(x)$ when the independent variable consists of a single term like x.

Thus $x^2, x^3, x^4, \ldots, x^n, \sqrt{x}, \sqrt[n]{x}, \sin x, \cos x, \tan x, \cot x, \sec x, \operatorname{cosec} x, \operatorname{vers} x, \sin^{-1}x, \cos^{-1}x, \tan^{-1}x, \cot^{-1}x, \sec^{-1}x, \operatorname{cosec}^{-1}x, \operatorname{vers}^{-1}x, a^x, \exp x, \cosh x, \sinh x, \tanh x, \ldots \log x, \cosh^{-1}x, \sinh^{-1}x, \tanh^{-1}x, \ldots$, all denoted generically by $\mathrm{f}x$, and most of them presumably familiar to the student, are simple functions of x, which we shall require hereafter.

We proceed to *differentiate* them, that is to find the *differential coefficient*, which is defined as follows:—

3. *Definition of a Differential Coefficient.*

If $\mathrm{f}x$ denotes any function of a variable quantity x, and if $\mathrm{f}(x+h)$ denotes the *same* function of $x+h$ when x receives a small *increment* h, then the *limiting value* of

$$\frac{\mathrm{f}(x+h)-\mathrm{f}x}{h},$$

when h is indefinitely diminished, is called the *differential coefficient* of $\mathrm{f}x$ with respect to x, and is denoted by

$$\frac{d\mathrm{f}x}{dx} \text{ or } \mathrm{f}'x.$$

This definition may be conveniently expressed as

$$\frac{d\mathrm{f}x}{dx} = \mathrm{lt}\frac{\mathrm{f}(x+h) - \mathrm{f}x}{h},$$

(lt) being the abbreviation employed to denote the limiting value, as h is indefinitely diminished and ultimately becomes zero.

Since $\mathrm{f}(x+h) - \mathrm{f}x$ is the increment of $\mathrm{f}x$ corresponding to the increment h of x, therefore

$$\mathrm{lt}\frac{\mathrm{f}(x+h) - \mathrm{f}x}{h}$$

is the *ultimate ratio* of the corresponding increments of $\mathrm{f}x$ and x, denoted by $d\mathrm{f}x$ and dx, and called the *differentials* of $\mathrm{f}x$ and x; and thus $\frac{d\mathrm{f}x}{dx}$ measures the *rate of increase* or *growth* of $\mathrm{f}x$; while $\frac{\mathrm{f}(x+h) - \mathrm{f}x}{h}$ represents the *average* rate of increase of $\mathrm{f}x$ from x to $x+h$.

The chief object at the outset of our subject is the determination of the differential coefficients of functions, and the application of them to the discussion of the geometrical and analytical properties of the functions.

The algebraical difficulty in the determination of the differential coefficient lies in the reduction of its original *indeterminate form* $\frac{0}{0}$ to a determinate limit.

(Hall and Knight, *Higher Algebra,* chap. xx.)

The name *derivative* or *derived function* is sometimes used instead of *differential coefficient.*

The differential coefficient $\mathrm{f}'x$ of a function of $\mathrm{f}x$ may be supposed to derive its name from being the coefficient which turns the differential of x into the differential of $\mathrm{f}x$.

(Wicksteed, *The Alphabet of Economic Science,* p. 32.)

4. *Differential Coefficient or Derivative of x^n.*

From the definition of a differential coefficient in § 3,

$$\frac{dx}{dx}=\text{lt}\frac{x+h-x}{h}=1.$$

$$\frac{dx^2}{dx}=\text{lt}\frac{(x+h)^2-x^2}{h}=\frac{0}{0}$$

$$=\text{lt}\frac{2xh+h^2}{h}=\frac{0}{0}\cdot$$

$$=\text{lt}(2x+h)=2x.$$

$$\frac{dx^3}{dx}=\text{lt}\frac{(x+h)^3-x^3}{h}=\frac{0}{0}$$

$$=\text{lt}\frac{3x^2h+3xh^2+h^3}{h}=\frac{0}{0}$$

$$=\text{lt}(3x^2+3xh+h^2)=3x^2.$$

$$\frac{dx^4}{dx}=\text{lt}\frac{(x+h)^4-x^4}{h}=\frac{0}{0}$$

$$=\text{lt}\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h}=\frac{0}{0}$$

$$=\text{lt}(4x^3+6x^2h+4xh^2+h^3)=4x^3.$$

And generally, when n is a positive integer,

$$\frac{dx^n}{dx}=\text{lt}\frac{(x+h)^n-x^n}{h}=\frac{0}{0}$$

(and expanding by the Binomial Theorem)

$$=\text{lt}\frac{nx^{n-1}h+\frac{n(n-1)}{1.2}x^{n-2}h^2+\ldots+h^n}{h}=\frac{0}{0}$$

$$=\text{lt}\left\{nx^{n-1}+\frac{n(n-1)}{1.2}x^{n-2}h+\ldots+h^{n-1}\right\}=nx^{n-1}.$$

By assuming the *convergency* of the Binomial Theorem we can make the same proof hold for showing that the differential coefficient of x^n with respect to x is nx^{n-1}, where n represents any constant number, whether integral, fractional, positive or negative.

But without using the Binomial Theorem or assuming its convergency, a proof may be given as follows:—

(i.) Suppose n a positive integer, and denote $x+h$ by x_1; then

$$\frac{dx^n}{dx}=\mathrm{lt}\frac{x_1{}^n-x^n}{x_1-x}=\frac{0}{0}\cdot$$

(and then by Division)

$$=\mathrm{lt}(x_1{}^{n-1}+xx_1{}^{n-2}+\ldots+x^{n-2}x_1+x^{n-1})=nx^{n-1},$$

when $h=0$, and therefore $x_1=x$.

(ii.) Suppose n a positive fraction p/q, where p and q are positive integers; and put $x=z^q$, $x_1=z_1{}^q$; then

$$\frac{dx^n}{dx}=\mathrm{lt}\frac{x_1{}^{p/q}-x^{p/q}}{x_1-x}$$

$$=\mathrm{lt}\frac{z_1{}^p-z^p}{z_1{}^q-z^q}$$

(dividing out z_1-z from numerator and denominator)

$$=\mathrm{lt}\frac{z_1{}^{p-1}+z_1{}^{p-2}z+\ \ldots\ +z_1z^{p-2}+z^{p-1}}{z_1{}^{q-1}+z_1{}^{q-2}z+\ \ldots\ +z_1z^{q-2}+z^{q-1}}$$

$$=\frac{pz^{p-1}}{qz^{q-1}}=\frac{p}{q}z^{p-q}=nx^{n-1}.$$

(iii.) Suppose n any negative number $-m$; then

$$\frac{dx^n}{dx}=\mathrm{lt}\frac{x_1{}^{-m}-x^{-m}}{x_1-x}=-\mathrm{lt}\frac{x_1{}^m-x^m}{x_1-x}\,\frac{1}{x_1{}^mx^m}$$

$$=-\ \frac{mx^{m-1}}{x^{2m}}=-mx^{-m-1}=nx^{n-1};$$

so that $\dfrac{dx^n}{dx}=nx^{n-1}$, universally, where n denotes any constant quantity.

Examples.—1. Determine from the definition the d.c. (differential coefficient) with respect to x of

$$\sqrt{x},\ x^{\frac{1}{3}},\ x^{\frac{2}{3}},\ x^{\frac{1}{4}},\ x^{\frac{3}{4}},\ x^{\frac{5}{4}},\ \frac{1}{\sqrt{x}},\ \frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3},\ \frac{1}{x^n},\ (x+a)^n,$$

$$\left(\frac{x}{n}\right)^n,\ \frac{2x+3}{x+1},\ \frac{ax+b}{Ax+B},\ a+bx+cx^2+\dots\ .$$

2. Give a rigorous proof that the differential coefficient of $x^{\sqrt{2}}$ with respect to x is $\sqrt{2}\ x^{\sqrt{2}-1}$.

5. *Geometrical Interpretation of Differentiation.*

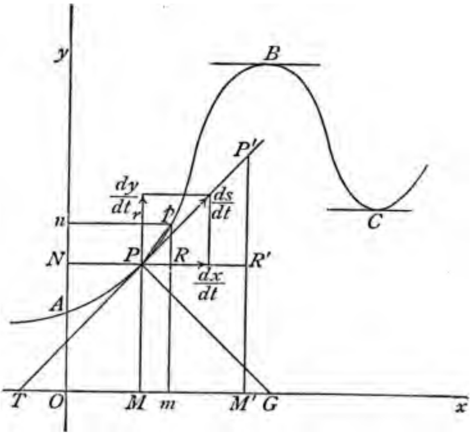


Fig.1

In Figure 1 the coordinates of a point P, referred to axes Ox and Oy at right angles, are denoted by x and y, where $OM=x$, $MP=y$; and in general, in the vertical elevation of an object, x is measured positive to the right, and negative to the left, y is measured positive upwards, and negative downwards; or, on a map or plan, Ox is drawn to the east, and Oy to the north.

Then the equation $y=\mathrm{f}x$ represents some curve *APB*, the assemblage of points whose coordinates satisfy this equation; so that if $OM=x$, then $MP=\mathrm{f}x$; the curve *APB* is now called the *graph* of the function fx (Chrystal, Algebra I., chap. xv.).

If $\qquad Mm=h$, then $Om=x+h$,

and $\qquad mp=\mathrm{f}(x+h)$, $Rp=\mathrm{f}(x+h)-\mathrm{f}x$;

so that $\qquad \dfrac{\mathrm{f}(x+h)-\mathrm{f}x}{h}=\dfrac{Rp}{PR}=\tan RPp.$

Now if the angle xTP is denoted by ψ, then since the direction of the tangent *TP* is the ultimate direction of the chord *Pp* when the point p has approached indefinitely near to *P*, therefore

$$\tan\psi=\mathrm{lt}\tan RPp=\mathrm{lt}\frac{Rp}{PR}$$

$$=\mathrm{lt}\frac{\mathrm{f}(x+h)-\mathrm{f}x}{h}=\frac{d\mathrm{f}x}{dx},\text{ or }\mathrm{f}'x,\text{ or }\frac{dy}{dx},\text{ or }dy/dx,$$

since $\qquad y=\ \mathrm{f}x.$

By giving x a *decrement* h as well as an increment h, we may define a differential coefficient by the relation

$$\frac{d\mathrm{f}x}{dx}=\mathrm{lt}\frac{\mathrm{f}(x+h)-\mathrm{f}(x-h)}{2h},$$

which is sometimes more elegant and useful than the definition in § 3, to which it is ultimately equivalent.

Figure 1 shows that dy/dx or $\tan\psi$ is positive if y increases with x, but that dy/dx is negative if y diminishes as x increases; and thus dy/dx is zero at the *turning points*, such as B or C, where y from increasing begins to diminish, or from diminishing begins to increase.

At B, y is said to have a *maximum* value, and at C a *minimum* value; and in each case $dy/dx=0$.

Thus to discover the maximum or minimum value of y, a function of x, we must first find the values of x which make $dy/dx=0$.

Afterwards we must determine whether, as x increases continually, dy/dx changes from positive to negative, in which case y has a *maximum* value; or whether dy/dx changes from negative to positive, in which case y has a *minimum* value; we shall return to this subject hereafter, although the present application to simple cases is easy.

Thus $ax-x^2$ has a maximum value $\frac{1}{4}a^2$, when $x=\frac{1}{2}a$ (fig. 33); and $2x^3-9x^2+12x-3$ has a maximum value 2 when $x=1$, and a minimum value 1 when $x=2$ (fig. 29).

Imagine the curve ABC to be a road, and dx a small horizontal forward step, dy the consequent vertical step; then dy and therefore dy/dx is positive if the traveller is ascending, and dy and dy/dx is negative if the traveller is descending, supposing dx always positive.

At the turning points, such as the top of a hill or the bottom of a valley, where y is a maximum or minimum, $dy/dx=0$.

The road is steepest where dy/dx has its maximum numerical value, and then the differential coefficient of dy/dx will be zero.

We may call $\tan\psi$ or dy/dx the *gradient* of the road, the ascent being dy/dx feet vertical for one foot horizontal; or otherwise expressed, as on railway plans, the gradient is 1 in $\cot\psi$ or dx/dy, meaning 1 foot vertical for dx/dy feet horizontal.

6. *The Tangent and Normal of a Curve.*

If x', y' are the coordinates of any point P' on the tangent TPP' to the curve APB at P (fig. 1), then

$$\frac{y'-y}{x'-x}=\frac{dy}{dx}, \quad \text{or} \quad y'-y=\frac{dy}{dx}(x'-x),$$

the *equation* of the tangent TP.

TM is called the *subtangent*, and MG the *subnormal* at P; PG being the *normal* at P, drawn perpendicular to the tangent through P. Therefore

$$TM=y\cot\psi=ydx/dy\,;\ MG=y\tan\psi=ydy/dx.$$

If T is to the right of M, $\tan\psi$ is negative, and

$$MT=-ydx/dy,\ GM=-ydx/dy.$$

Also the equation of the normal GP at P is

$$y'-y=-\frac{dx}{dy}(x'-x).$$

Let us apply these principles to some simple curves, the graphs of (i.) $y=x^2$, (ii.) $y=x^{\frac{1}{2}}$, (iii.) $y=x^{-1}$, (iv.) $y^2=x^3$.

(i.) $y=x^2$, a *parabola* (fig. 2 i.).

Here $dy/dx=2x$; so that the equation of the tangent is $y'-y=2x(x'-x)$; which, since $y=x^2$, may be written

$$2xx'=y'+y, \text{or } \frac{x'}{\frac{1}{2}x}+\frac{y'}{-y}=1.$$

Thus $OT=\frac{1}{2}x$, $TM=\frac{1}{2}x$, and therefore T bisects OM.

The equation of the normal is $\dfrac{x'}{2x^3+x}+\dfrac{y'}{y+\frac{1}{2}}=1.$

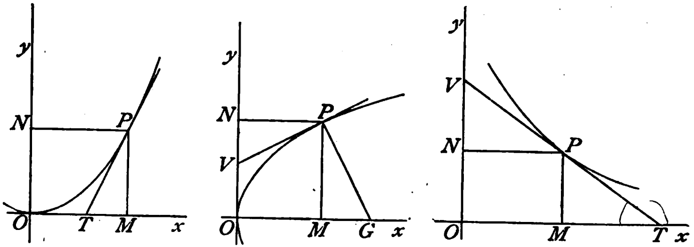
(ii.) $y=x^{\frac{1}{2}}$, another *parabola* (fig. 2 ii.).

Here $$dy/dx=\tfrac{1}{2}x^{-\frac{1}{2}}=\tfrac{1}{2}y/x\,;$$

so that the equations of the tangent and normal become

$$\frac{y'}{\frac{1}{2}y}-\frac{x'}{x}=1\,; \quad \frac{x'}{x+\frac{1}{2}}+\frac{y'}{2y^3+y}=1\,;$$

thus $TO=OM$, $OV=VN$, and $MG=\frac{1}{2}$.



(i) (ii) (iii)

Fig.2

(iii.) $y=x^{-1}$, a *hyperbola* (fig. 2 iii.).

Here $$\frac{dy}{dx}=-x^{-2}=-\frac{1}{x^2}=-\frac{y}{x};$$
and the equation of the tangent becomes
$$\frac{x'}{2x}+\frac{y'}{2y}=1,$$
so that $$OT=2x,\ OV=2y.$$

The equation of the normal is
$$xx'-yy'=x^2-y^2.$$

(iv.) $y^2=x^3$, or $y=x^{\frac{3}{2}}$, a curve called the *semicubical parabola*, and something like figure (i.).

Here $dy/dx=\frac{3}{2}x^{\frac{1}{2}}$, so that $TM=\frac{2}{3}x$, $OT=\frac{1}{3}x=\frac{1}{3}OM$.

(v.) Prove that in the curve $y^m=cx^n$, the equations of the tangent and normal are
$$\frac{x'}{mx}-\frac{y'}{ny}=\frac{1}{m}-\frac{1}{n},\quad mxx'+nyy'=mx^2+ny^2.$$

* 7. Supposing h again, as at first, to denote a finite increment of x, and supposing that neither fx nor f$'x$ becomes infinite between P and p, and that they change gradually between these points; then in the graph of fx

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*Articles which may be omitted at a first reading are marked with an asterisk * the tangent TK at some point K between P and p (fig. 3) is parallel to the chord Pp; and

$$\frac{\mathrm{f}(x+h)-\mathrm{f}x}{h}=\tan RPp=\tan xTK=\mathrm{f}'(x+\theta h),$$

where $x+\theta h=OH$, the abscissa of K; and then $\theta=MH/Mm$, a proper fraction, which is some unknown function of x and h.

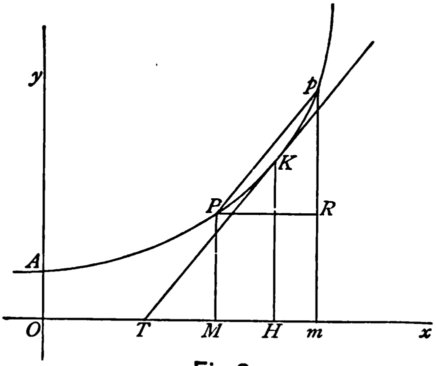


Fig.3

Therefore $\mathrm{f}(x+h)=\mathrm{f}x+h\mathrm{f}'(x+\theta h)$,
a theorem required subsequently in Chapter IV.

This leads to another definition of $\mathrm{f}'x$, employed by Lagrange, according to which $\mathrm{f}'x$ is defined as the coefficient of h in the algebraical expansion of $\mathrm{f}(x+h)$ in ascending positive integral powers of h.

To illustrate geometrically exceptional cases of $\mathrm{f}x$ and $\mathrm{f}'x$, consider $\mathrm{f}x=(x-b)/(x-a)$, or $\mathrm{f}x=(x-a)^{\frac{2}{3}}$, taking $x<a$, and $x+h>a$.

Given that $\mathrm{f}x=a+bx+cx^2$,
then $\theta=\frac{1}{2}$; and generally, when h is small, we shall find $\theta ≋ \frac{1}{2}$; the symbol ≋ denoting approximate equality.

If k denotes a different increment of x, then

$$\mathrm{f}(x+k)=\mathrm{f}x+k\mathrm{f}'(x+\phi k),$$

where ϕ is a proper fraction, the same function of x and k which θ is of x and h; so that

$$\frac{\mathrm{f}(x+h)-\mathrm{f}x}{\mathrm{f}(x+k)-\mathrm{f}x}=\frac{h}{k}\frac{\mathrm{f}'(x+\theta h)}{\mathrm{f}'(x+\phi k)}.$$

Ultimately, when h and k are sufficiently small, we may put

$$\frac{\mathrm{f}'(x+\theta h)}{\mathrm{f}'(x+\phi k)}=1,$$

without sensible error; and then

$$\frac{\mathrm{f}(x+h)-\mathrm{f}x}{\mathrm{f}(x+k)-\mathrm{f}x}=\frac{h}{k},$$

which is the rule of *proportional parts*, employed in Trigonometry; equivalent to neglecting the curvature of the arc Pp, and replacing the arc by a straight line.

Again supposing APp to represent the profile or elevation of a road, and $\mathrm{f}'x$ to represent the slope or gradient at P, $\mathrm{f}'(x+h)$ at p, then the gradient $\mathrm{f}'(x+\theta h)$ at K is the average gradient of the chord Pp, represented by $\{\mathrm{f}(x+h)-\mathrm{f}x\}/h$.

8. Instead of the letter h the symbol Δx is often employed to denote the finite increment of x; and Δy is then used to denote the corresponding increment of y, where y is a function of x, denoted by $\mathrm{f}x$.

Then, with this notation,

$$y+\Delta y=\mathrm{f}\ (x+\Delta x),$$

and $\Delta y=\mathrm{f}(x+\Delta x)-\mathrm{f}x$;

so that $\Delta y=Rp$, if $\Delta x=PR$ (fig. 1);

and $\dfrac{dy}{dx}=\mathrm{lt}\dfrac{\Delta y}{\Delta x}$.

In this notation, Δ (Delta) and d are not algebraical factors, but symbolic operations; so that Δx and dx must be considered as inseparable quantities, the x being qualified by the Δ or d, and not multiplied algebraically by it.

Suppose for instance it is found that the difference of range of a gun over the sea at high and at low water is Δx, when Δy represents the rise and fall of the tide; then the tangent of the angle of descent of the projectile into the water is $\Delta y/\Delta x$, approximately.

9. Again, let the length of the arc AP of the curve APp, measured from any fixed point A to the variable point P, be denoted by s (fig. 1).

Denoting by Δs the increment of s, corresponding to the increment Δx of x, then the arc $APp = s + \Delta s$, and the arc $Pp = \Delta s$.

Now, when the point p approaches to coincidence with P along the curve pP, it is assumed as axiomatic that

$$\text{lt}\frac{\text{chord } Pp}{\text{arc } Pp} = 1;$$

or
$$\text{lt}\frac{\sqrt{(PR^2 + Rp^2)}}{\text{arc } Pp} = 1;$$

or
$$\text{lt}\frac{\sqrt{(\Delta x^2 + \Delta y^2)}}{\Delta s} = 1;$$

where Δx^2 means $(\Delta x)^2$;

or
$$\text{lt}\left(\frac{\Delta x^2}{\Delta s^2} + \frac{\Delta y^2}{\Delta s^2}\right) = 1.$$

Therefore
$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} = 1,$$

or, as it is sometimes written,

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1.$$

The assumption, which has been taken as axiomatic, that lt(chord *Pp*)/(arc *Pp*) = 1, is proved in Newton's Lemma VII. (*Principia*, Book I., § 1), as follows—

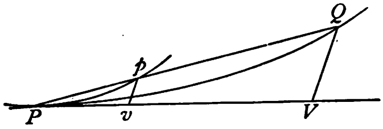


Fig.4

As the point *p* approaches to the point *P*, let the chord *Pp*, and the tangent *Pv* be produced to points *Q*, *V*, at a finite distance, where *QV* is parallel to *pv*, and the arc *PQ* is similar to the arc *Pp*; so that *PQV* may be considered the magnified image of the similar microscopic figure *Ppv*.

Now let the point *p* move on the curve *pP* close up to *P*, and let *pv* move parallel to itself up to *P*; then the angle *QPV* always diminishes and ultimately vanishes; and the chord *PQ*, the arc *PQ* and the tangent *PV* are ultimately coincident; therefore also the chord *Pp*, the arc *Pp* and the tangent *Pv*, when vanishing together, will have a ratio of equality.

Again (fig. 1) $\cos\psi = \cos xTP = \operatorname{lt}\cos RPp = \operatorname{lt}\dfrac{PR}{Pp}$

$$= \operatorname{lt}\frac{\Delta x}{\Delta s}\operatorname{lt}\frac{\text{arc } Pp}{\text{chord } Pp} = \frac{dx}{ds};$$

and $\sin\psi = \sin xTP = \operatorname{lt}\sin RPp = \operatorname{lt}\dfrac{RP}{Pp}$

$$= \operatorname{lt}\frac{\Delta y}{\Delta s}\operatorname{lt}\frac{\text{arc } Pp}{\text{chord } Pp} = \frac{dy}{ds}.$$

Since $(\cos\psi)^2 + (\sin\psi)^2 = 1,$

therefore $\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2 = 1$, as before.

The tangent $TP = y \operatorname{cosec} \psi = y ds/dy$; and the normal $PG = y \sec \psi = y ds/dx$; and the length of the perpendicular p from the origin O upon the tangent at P is

$$p = x \sin \psi - y \cos \psi = x\frac{dy}{ds} - y\frac{dx}{ds}.$$

Since $\qquad (\sec \psi)^2 = 1 + (\tan \psi)^2,$

and $\qquad \sec \psi = ds/dx,\ \tan \psi = dy/dx;$

therefore $\qquad \dfrac{ds^2}{dx^2} = 1 + \dfrac{dy^2}{dx^2}.$

Similarly, from $(\operatorname{cosec} \psi)^2 = (\cot \psi)^2 + 1$,

we deduce $\qquad \dfrac{ds^2}{dy^2} = \dfrac{dx^2}{dy^2} + 1.$

Sometimes the gradient is measured by the vertical rise for one foot on the incline, that is by dy/ds or the sine of the slope ψ; and then the gradient is one in ds/dy or cosec ψ.

We may say that the gradient is measured by the tangent of the slope on the plan, and by the sine of the slope on the constructed work; in the inclines of roads and railways the slope is sufficiently small for no difference to be perceptible in the two methods.

10. *Velocity expressed by a Differential Coefficient.*

If x and y, the coordinates of a point P, moving along the curve APp (fig. 1); the path of a projectile in the air or of a train on a railway, for instance) are given as functions of the time t; and if Δt denotes the time occupied in moving from P to p, a distance Δs; then the *average* velocity from P to p is $\Delta s/\Delta t$; and the actual velocity of P, in the direction of the tangent TP, is

$$\operatorname{lt}\frac{\Delta s}{\Delta t} = \frac{ds}{dt},$$

the rate of increase or growth of s per unit of t.

The component velocity of P parallel to Ox is the velocity of M; and treating M as a point moving in the straight line Ox, the *average* velocity from M to m is $\Delta x/\Delta t$; and therefore the actual velocity of M

$$=\operatorname{lt}\frac{\Delta x}{\Delta t}=\frac{dx}{dt},$$

the rate of growth of x per unit of t.

Similarly the component velocity of P parallel to Oy, or the rate of growth of y is dy/dt, the velocity of N.

Since the resultant velocity of P, ds/dt, has the components dx/dt and dy/dt in two rectangular directions, therefore

$$\frac{ds^2}{dt^2}=\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}.$$

Making $t=s$, x, or y, we obtain the previous results.

According to Newton's original definition, the *fluxion* of a variable quantity x is its velocity or rate of growth, and was denoted by $\dot{x}$; in Leibnitz's notation of the Differential Calculus it is represented by $\frac{dx}{dt}$ or dx/dt; thus differentiation with respect to the time t may be considered the typical fundamental conception of this Calculus, which investigates the rate at which variable quantities change, now not only with respect to the time t, but also with respect to any other independent variable.

11. *Differentiation of a Function of a Function.*

Since

$$\frac{\Delta y}{\Delta t}=\frac{\Delta y}{\Delta x}\frac{\Delta x}{\Delta t},$$

therefore, proceeding to the limit,

$$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt},$$

where x is a function of t, and y is a function of x, and therefore of t.

Thus, with the notation of §§ 9, 10, (fig. 1),

$$\frac{dx}{dt}=\frac{dx}{ds}\frac{ds}{dt}=\cos\psi\frac{ds}{dt},\quad \frac{dy}{dt}=\frac{dy}{ds}\frac{ds}{dt}=\sin\psi\frac{ds}{dt};$$

showing how the component velocities are obtained by resolving the resultant velocity.

Denoting the velocity ds/dt by v, then dv/dt, the rate of growth of v, is called the *acceleration;* and

$$\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=\frac{dv}{ds}v=\frac{d\frac{1}{2}v^2}{ds}.$$

With different letters, supposing z is a function of y, and y is a function of the independent variable x; and supposing the increment Δx of x gives y the increment Δy, and z the increment Δz; then since, algebraically,

$$\frac{\Delta z}{\Delta x}=\frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x},$$

therefore

$$\frac{dz}{dx}=\frac{dz}{dy}\frac{dy}{dx},$$

the rule for the *differentiation of a function of a function.*

Thus, suppose $z=y^m$, where y is a function of x; then

$$\frac{dz}{dx}=\frac{dy^m}{dx}=\frac{dy^m}{dy}\frac{dy}{dx}=my^{m-1}\frac{dy}{dx};$$

or, if $y=\mathrm{f}x$, then $\dfrac{d(\mathrm{f}x)^m}{dx}=m(\mathrm{f}x)^{m-1}\mathrm{f}'x.$

Putting $z=Fy$ and $y=\mathrm{f}x$, so that $z=F(\mathrm{f}x)$; then

$$\frac{dz}{dx}=\frac{dz}{dy}\frac{dy}{dx}=F'y\ \mathrm{f}'x=F'(\mathrm{f}x)\ \mathrm{f}'x.$$

Thus if $z=(a+bx^n)^m$, $\dfrac{dz}{dx}=m(a+bx^n)^{m-1}nbx^{n-1}.$

With $y=\mathrm{f}x$, then $dy/dt=\mathrm{f}'x\,.\,dx/dt$; and this, in the notation of *Differentials,* is abbreviated to $dy=\mathrm{f}'x\,.\,dx$, in which the independent variable is left arbitrary; and dx, dy are called the differentials of x and y, as in § 3.

12. *Differentiation of the Sum, Product, and Quotient of Functions.*

Let u and v denote any two functions of x; and let Δu and Δv denote the increments of u and v, corresponding to the increment Δx of x; then (i.) for their sum,

$$\frac{d(u+v)}{dx}=\text{lt}\frac{\Delta u+\Delta v}{\Delta x}=\text{lt}\left(\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}\right)=\frac{du}{dx}+\frac{dv}{dx};$$

so that the *d.c. of the sum of two functions is the sum of the d.c.'s of the functions.*

Similarly, for the difference of u and v,

$$\frac{d(u-v)}{dx}=\frac{du}{dx}-\frac{dv}{dx}.$$

And generally, if a and b denote constant factors,

$$\frac{d(au+bv)}{dx}=a\frac{du}{dx}+b\frac{dv}{dx}.$$

Thus $\frac{d}{dx}(ax^m+bx^n-cx^p+\ldots)$

$$=amx^{m-1}+bnx^{n-1}-cpx^{p-1}+\ldots;$$

and if $s=ut+\frac{1}{2}ft^2$, $v=ds/dt=u+ft$.

We notice now that the differential coefficient of a *constant* is zero; as is obvious, since a constant does not change with the independent variable.

(ii.) For the product of u and v,

$$\frac{duv}{dx}=\text{lt}\frac{(u+\Delta u)(v+\Delta v)-uv}{\Delta x}$$

$$=\text{lt}\frac{\Delta u}{\Delta x}v+\text{lt}(u+\Delta u)\frac{\Delta v}{\Delta x}=\frac{du}{dx}v+u\frac{dv}{dx};$$

because $\text{lt}\frac{\Delta u}{\Delta x}=\frac{du}{dx}$, $\text{lt}\frac{\Delta v}{\Delta x}=\frac{dv}{dx}$, and $\text{lt}(u+\Delta u)=u$.

Hence the rule for differentiating a product: *Differentiate with respect to each factor and add the results.*

For instance if $u=(x-a)^m$, $v=(x-b)^n$,

$$\frac{d(x-a)^m(x-b)^n}{dx}=m(x-a)^{m-1}(x-b)^n+(x-a)^m n(x-b)^{n-1}$$
$$=(x-a)^{m-1}(x-b)^{n-1}\{(m+n)x-mb-na\}.$$

It is important for subsequent purposes not to alter the order of the factors in differentiating; thus for a product of three factors u, v, w,

$$\frac{duvw}{dx}=\frac{du}{dx}vw+u\frac{dvw}{dx}=\frac{du}{dx}vw+u\frac{dv}{dx}w+uv\frac{dw}{dx};$$

and so on for any number of factors.

To illustrate geometrically, refer to fig. 1, in which P is a point moving along the curve AP, whose coordinates are x, y at the time t.

In the time Δt, P has moved to p, whose coordinates are $x+\Delta x$, $y+\Delta y$; and the rectangle $OMPN=xy$ has increased by the rectangles $Pm=\Delta x.y$, and $nR=(x+\Delta x)\Delta y$; and the formula

$$\frac{dxy}{dt}=\frac{dx}{dt}y+x\frac{dy}{dt}$$

expresses the rate of growth of the rectangle xy in the two directions Ox and Oy.

(iii.) For u/v, the quotient of u by v,

$$\frac{d\frac{u}{v}}{dx}=\text{lt}\frac{\frac{u+\Delta u}{v+\Delta v}-\frac{u}{v}}{\Delta x}=\text{lt}\frac{(u+\Delta u)v-u(v+\Delta v)}{\Delta x(v+\Delta v)v}$$
$$=\text{lt}\frac{\frac{\Delta u}{\Delta x}v-u\frac{\Delta v}{\Delta x}}{(v+v\Delta)v}=\frac{\frac{du}{dx}v-u\frac{dv}{dx}}{v^2};$$

hence the rule for differentiating a fraction: *the differential coefficient is equal to*

$$\frac{\textit{d.c. of num}^r.\times\textit{denom}^r.-\textit{num}^r.\times\textit{d.c. of denom}^r.}{(\textit{denominator})^2}.$$

Thus if $y=\dfrac{3x-x^3}{1-3x^2}$

$$\frac{dy}{dx}=\frac{(3-3x^2)(1-3x^2)-(3x-x^3)(-6x)}{(1-3x^2)^2}=3\left(\frac{1+x^2}{1-3x^2}\right)^2.$$

If the denominator of the fraction is of the form v^n, for instance, if $y=u/v^n$,

then $$\frac{dy}{dx}=\frac{\dfrac{du}{dx}v^n-unv^{n-1}\dfrac{dv}{dx}}{v^{2n}}=\frac{\dfrac{du}{dx}v-nu\dfrac{dv}{dx}}{v^{n+1}},$$

after removing the common factor v^{n-1} from the numerator and denominator.

Thus if $y=\dfrac{(x-a)^m}{(x-b)^n}$,

$$\frac{dy}{dx}=\frac{m(x-a)^{m-1}(x-b)^n-(x-a)^m n(x-b)^{n-1}}{(x-b)^{2n}}$$

$$=\frac{(x-a)^{m-1}\{(m-n)x-mb+na\}}{(x-b)^{n+1}}.$$

In such cases it is sometimes preferable to write the fraction in the form of a product; then $y=uv^{-n}$;

and $$\frac{dy}{dx}=\frac{du}{dx}v^{-n}-unv^{-n-1}\frac{dv}{dx}$$

$$=\frac{\dfrac{du}{dx}v-nu\dfrac{dv}{dx}}{v^{n+1}},$$

as before, but without introducing extraneous factors.

Thus if $y=\dfrac{1}{(x-a)^m(x-b)^n}$,

writing it in the form

$$y=(x-a)^{-m}(x-b)^{-n},$$

then $$\frac{dy}{dx}=-m(x-a)^{-m-1}(x-b)^{-n}-(x-a)^{-m}n(x-b)^{-n-1}$$

$$=-\frac{(m+n)x-na-mb}{(x-a)^{m+1}(x-b)^{n+1}}.$$

Examples of Differentiation.

(1) Prove the following differentiations

(i.) $y=\sqrt{(2x)},\ \frac{dy}{dx}=\frac{1}{\sqrt{(2x)}}.$

(*Solution.*—Here $y=(2x)^{\frac{1}{2}}$, and therefore

$$\frac{dy}{dx}=\tfrac{1}{2}\times(2x)^{-\frac{1}{2}}\times 2=(2x)^{-\frac{1}{2}}=\frac{1}{\sqrt{(2x)}}.)$$

(ii.) $y=(nx)^{\frac{1}{n}},\ \frac{dy}{dx}=(nx)^{\frac{1}{n}-1}.$

(iii.) $y=\sqrt{(x+a)},\ \frac{dy}{dx}=\frac{1}{2\sqrt{(x+a)}}.$

(iv.) $y=\frac{1}{\sqrt{(a-x)}},\ \frac{dy}{dx}=\frac{1}{2(a-x)^{\frac{3}{2}}}.$

(v.) $y=\sqrt{(a^2+x^2)},\ \frac{dy}{dx}=\frac{x}{\sqrt{(a^2+x^2)}}.$

(vi.) $y=\frac{1}{\sqrt{(a^2-x^2)}},\ \frac{dy}{dx}=\frac{x}{(a^2-x^2)^{\frac{3}{2}}}.$

(vii.) $y=\frac{x}{\sqrt{(x^2+a^2)}},\ \frac{dy}{dx}=\frac{a^2}{(x^2+a^2)^{\frac{3}{2}}}.$

(*Solution.*—Employing the rule of § 12,

$$\frac{dy}{dx}=\frac{\sqrt{(x^2+a^2)}-x\dfrac{x}{\sqrt{(x^2+a^2)}}}{x^2+a^2}=\frac{a^2}{(x^2+a^2)^{\frac{3}{2}}}.)$$

(viii.) $y=\sqrt{(2ax-x^2)},\ \frac{dy}{dx}=\frac{a-x}{\sqrt{(2ax-x^2)}}.$

(2) Differentiate with respect to x,

$\frac{1}{7}x^7-\frac{3}{5}a^2x^5+a^4x^3-a^6x,\ x\sqrt{(1-x^2)},\ x(1-2x^2)\sqrt{(1-x^2)},$

$\frac{a-x}{a+x},\ \sqrt{\frac{a-x}{a+x}},\ \frac{2x}{1-x^2},\ \frac{3x+x^3}{1+3x^2},\ \frac{4x-4x^3}{1-6x^2+x^4},\ \frac{5x+10x^3+x^5}{1+10x^2+5x^4}.$

(3) Deduce the rule for the differentiation of a quotient from the rule for the differentiation of a product.

13. *Implicit and Explicit Relations between two variables, x and y.*

When the variables x and y are connected together by any relation, for instance by

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

the general equation of a conic section, or by

$$x^3-3axy+y^3=0,$$

the equation of a *cubic* curve, this relation is called an *implicit relation* between x and y, and is denoted generically by the notation

$$\mathrm{F}(x, y)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{A}).$$

When however it is possible by solution of the equation to obtain y in terms of x, or x in terms of y, then y is called an *explicit function* of x, or x of y, denoted generically as before by $y=\mathrm{f}x$, or $x=\phi y$ (in the notation of *inverse* functions, § 25, $x=\mathrm{f}^{-1}y$).

In the above implicit relations between x and y, we cannot obtain y explicitly in terms of x, or x in terms of y, except by the solution of a quadratic or cubic equation; but, for instance, from the implicit relation

$$x^2y^2-a^2(x^2-y^2)=0,$$

the equation of a *quartic* curve, we obtain *explicitly*, by mere transposition,

$$y^2=\frac{a^2x^2}{a^2+x^2}, \quad \text{or } x^2=\frac{a^2y^2}{a^2-y^2}.$$

From an implicit relation (A) between x and y to find dy/dx, it is not necessary to express y as an explicit function of x; but we differentiate (A) with respect to x, treating y as a function of x, and obtain what is called the *first derived equation* and thence determine dy/dx as a function of x and y, without solving the equation (A);

thus from the implicit relation or equation of a conic section we obtain the first derived equation.

$$2ax+2hy+2hx\frac{dy}{dx}+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0,$$

so that $$\frac{dy}{dx}=-\frac{ax+hy+g}{hx+by+f}.$$

For changing the coordinates x, y of a point on the conic section to $x+\Delta x$, $y+\Delta y$, the coordinates of an adjacent point on the curve, then

$$\begin{aligned}&\mathrm{F}(x+\Delta x,\ y+\Delta y)=\\ &a(x+\Delta x)^2+2h(x+\Delta x)(y+\Delta y)+b(y+\Delta y)^2\\ &\qquad+2g(x+\Delta x)+2f(y+\Delta y)+c=0;\end{aligned}$$

and therefore, by subtracting $F(x, y)=0$,

$$\begin{aligned}&a(2x\Delta x+\Delta x^2)+2h(\Delta x\,.\,y+x\Delta y+\Delta x\Delta y)\\ &\qquad+b(2y\Delta y+\Delta y^2)+2g\Delta x+2f\Delta y=0;\end{aligned}$$

and dividing by Δx,

$$\begin{aligned}&a(2x+\Delta x)+2h(y+x\frac{\Delta y}{\Delta x}+\Delta y)+b(2y+\Delta y)\frac{\Delta y}{\Delta x}\\ &\qquad+2g+2f\frac{\Delta y}{\Delta x}=0;\end{aligned}$$

which in the limit, when Δx and Δy are indefinitely small, becomes the first derived equation.

Again, from the implicit relation

$$x^3-3axy+y^3=0,$$

we obtain the first derived equation

$$3x^2-3ay-3ax\frac{dy}{dx}+3y^2\frac{dy}{dx}=0,$$

so that $$\frac{dy}{dx}=\frac{x^2-ay}{ax-y^2}.$$

Examples.—Find dy/dx from the implicit relations

(1) $x^2+y^2=c^2$.

(2) $(x-a)^2+(y-b)^2=c^2$.

(3) $\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1$.

(4) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

(5) $x^m y^n=a^{m+n}$.

(6) $x^5+y^5-5a^2x^2y=0$.

(7) Given the implicit relation

$$x^3-3axy+y^3=0,$$

prove that $dy/dx=0$, when $x=a\sqrt[3]{2}$, $y=a\sqrt[3]{4}$;

$dx/dy=0$, when $x=a\sqrt[3]{4}$, $y=a\sqrt{2}$.

(8) Find where $dy/dx=0$ and where $dx/dy=0$, in the curve $x^5+y^5-5a^2x^2y=0$.

14. *Tracing Curves.*

The student should at this stage be exercised in tracing or plotting simple curves from their equations, thus exhibiting to the eye the flow or march of the function $y=\mathrm{f}x$, or the *graph* of the function $\mathrm{f}x$ (§ 5).

Most of these curves should be already familiar to the student who has studied the elements of Analytical Geometry, as given in Smith's *Conic Sections*.

The curves should be drawn to scale as carefully as possible; for this purpose the paper which can be bought ruled into small squares is useful.

By drawing these curves and interpreting geometrically the differential coefficients obtained from their equations, the student will begin to understand the use of the Calculus, always difficult to explain to the beginner.

Examples.—Draw the curves:

(1) $y=1, x, x^2, x^3, x^4, \sqrt{x}, \frac{1}{x}, \frac{1}{x^2}, \ldots$ (on one diagram).

(2) $y=x-x^2, x-x^3, x+x^3, x+\frac{1}{x}, \ldots$

(3) $y^2 = 1,\ x,\ x^2,\ x^3,\ x^4,\ \sqrt{x},\ \frac{1}{x},\ \frac{1}{x^2}, \ldots$.(on one diagram).

(4) $y^2 = 1 - x^2,\ x^2 - 1,\ x^2 + 1,\ x - x^2,\ x + x^2,\ x^2 - x, \ldots$.

(5) $x + y = 1,\ x^2 + y^2 = 1,\ x^3 + y^3 = 1,\ x^4 + y^4 = 1,\ \sqrt{x} + \sqrt{y} = 1,$
$\frac{1}{x} + \frac{1}{y} = 1,\ \frac{1}{x^2} + \frac{1}{y^2} = 1, \ldots$.

(6) $x^2 - y^2 = 1,\ y^2 - x^2 = 1,\ \frac{1}{x^2} - \frac{1}{y^2} = 1,\ \frac{1}{y^2} - \frac{1}{x^2} = 1.$

(7) $y = \frac{x-1}{x-2},\ \frac{(x-1)(x-2)}{x-3},\ \frac{(x-1)(x-2)}{(x-3)(x-4)},\ \frac{(x-1)(x-3)}{(x-2)(x-4)}, \ldots$.

(8) $x^2 - 4x + 2y = 0,\ xy - 2x - y = 0,\ (y-x)^2 = 1 - x^2.$

(9) Prove the properties of the following curves and sketch the curves:—

(i.) In $y^2 = px$, a parabola, the subnormal MG is constant.

(ii.) In $x^2 + y^2 = a^2$, a circle, the normal PG is constant.

(iii.) In $x^2 + y^2 = 2ay$, a circle, $OT = TP$.

(iv.) In $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, the part of the tangent TV intercepted by the axes is of constant length.

(v.) In $by^2 = x^3$, a semicubical parabola,
$TM^2 = \frac{8}{27}b \,.\, MG$; and generally in $y^m = cx^n$,
MG varies as $(TM)^{(2m-n)/n}$.

(vi.) In $\frac{1}{x^n} + \frac{1}{y^n} = \frac{1}{a^n}$, $OT = \frac{x^{n+1}}{a^n}$; and determine p, the perpendicular from O upon the tangent.

(10) Prove that the equation of the tangent at (xy) of

(i.) The *circle* (fig. 13) $x^2 + y^2 = c^2$ is $xx' + yy' = c^2$; or of the *circle* $(x-a)^2 + (y-b)^2 = c^2$ is

$$(x-a)(x'-a) + (y-b)(y'-b) = c^2;$$

(ii.) The *ellipse* (fig. 8) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$;

(*Solution*:—Forming the first derived equation,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0, \text{ so that } \frac{dy}{dx}=-\frac{b^2x}{a^2y};$$

and then the equation of the tangent TPV is

$$y'-y=-\frac{b^2x}{a^2y}(x'-x), \quad \text{or} \quad \frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.)$$

(iii.) The *hyperbola* (fig. 14) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{xx'}{a^2}-\frac{yy'}{b^2}=1$.

(iv.) The *parabola* (fig. 2 ii.) $y^2=px$ is $\frac{y'}{\frac{1}{2}y}-\frac{x'}{x}=1$.

(v.) The *hyperbola* (fig. 2 iii.) $xy=c^2$ is $\frac{x'}{2x}+\frac{y'}{2y}=1$.

(This is called the *isothermal curve*, as it exhibits the relation between x the volume and y the pressure of a given quantity of gas at a constant temperature.)

(vi.) $x^my^n=a^{m+n}$ is $\frac{x'}{(1+n/m)x}+\frac{y'}{(1+m/n)y}=1$.

(This is called the *adiabatic* or *steam curve*, when we put for instance $m=7$ and $n=5$, or $m=10$ and $n=9$.)

(vii.) $\left(\frac{x}{a}\right)^m=\left(\frac{y}{b}\right)^n$ is $\frac{x'}{(1-n/m)x}+\frac{y'}{(1-m/n)y}=1$.

Determine also the equation of the normal of these curves.

(11) Prove that the curves $x^my^n=a^{m+n}$ and $\frac{x^2}{m}-\frac{y^2}{n}=b^2$ cut at right angles.

(12) Prove that the equation of the tangent to the parabola $\sqrt{x}+\sqrt{y}=\sqrt{a}$ is $\frac{x'}{\sqrt{x}}+\frac{y'}{\sqrt{y}}=\sqrt{a}$, and then $OT+OV=a$; and generally, the equation of the tangent of $x^n+y^n=a^n$ is $x^{n-1}x'+y^{n-1}y'=a^n$.

(12) Prove that if

(i.) $9ay^2=4x^3$, $\frac{ds}{dx}=\sqrt{\left(1+\frac{x}{a}\right)}$;

(ii.) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, $\frac{ds}{dx}=\left(\frac{a}{x}\right)^{\frac{1}{3}}$;

(iii.) $y^2=2lx$, $\frac{ds}{dy}=\sqrt{\left(1+\frac{y^2}{l^2}\right)}$.

(13) Prove that the conic section

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

can be described with component velocities

$$dx/dt=hx+by+f,\ dy/dt=-ax-hy-g,$$

or $dx/dt=\quad\sqrt{\{(h^2-ab)x^2+2(fh-bg)x+f^2-bc\}}$,

$$dy/dt=-\sqrt{\{(h^2-ab)y^2+2(gh-af)y+g^2-ac\}}.$$

Prove also that the equation of the tangent at (xy) is

$$(ax+hy+g)x'+(hx+by+f)y'+gx+fy+c=0.$$

15. *Algebraical and Transcendental Functions.*

So far we have dealt only with *algebraical* functions of x, an algebraical function being defined as one which is *composed of a finite number of the algebraical operations of* Addition, Subtraction, Multiplication, Division, Involution, *and* Evolution.

An algebraical function of x may be defined in the most general manner as *the root of an algebraical equation of integral degree, the coefficients of which are rational integral algebraical functions of x;* and a rational integral algebraical function of x is defined as *the sum of a finite number of terms, multiples of integral powers of x, such as*

$$ax^n+bx^{n-1}+cx^{n-2}+\dots,$$

where n is a positive integer; and then n, the highest power of x, is called the *degree* of the function.

But now we pass on to the functions of Trigonometry, such as $\sin x$, $\cos x$, ... , which are called *transcendental functions;* every function of x which is not an *algebraical* function being called a *transcendental* function.

Suppose we require to differentiate $\sin x$; then from the definition of a differential coefficient in § 3,

$$\frac{d\sin x}{dx}=\operatorname{lt}\frac{\sin(x+h)-\sin x}{h}=\operatorname{lt}\frac{2\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h}$$

$$=\operatorname{lt}\cos(x+\tfrac{1}{2}h)\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}=\cos x;$$

since $\operatorname{lt}\cos(x+\frac{1}{2}h)=\cos x$, and $\operatorname{lt}(\sin\frac{1}{2}h)/\frac{1}{2}h=1$, when x and therefore h is expressed in circular measure.

Or, more simply from the definition of § 5,

$$\frac{d\sin x}{dx}=\operatorname{lt}\frac{\sin(x+h)-\sin(x-h)}{2h}=\operatorname{lt}\cos x\frac{\sin h}{h}=\cos x;$$

and so on for the other functions of Trigonometry.

But if x is given in degrees, or minutes, or seconds, then

$$\frac{d\sin x^\circ}{dx}=\frac{\pi}{180}\cos x^\circ,\ \frac{d\sin x'}{dx}=\frac{\pi}{180\times 60}\cos x',$$

$$\frac{d\sin x''}{dx}=\frac{\pi}{180\times 60\times 60}\cos x'';$$

so that extraneous factors are avoided by the use of circular measure.

We thus require a knowledge of the circular measure of an angle and of the formulas of Trigonometry, a subject with which the student is presumably already familiar, but on which it will be desirable to make a slight digression, so far as is required for our purposes.

16. *Trigonometry. The Circular Measure, and the Circular Functions of an Angle.*

Definitions.—The circular measure (θ) of an angle AOP is the number giving the ratio of the circular arc AP to the radius OA.

The unit angle is the angle subtended by an arc equal to its radius; this angle is called the *radian*, and contains

$$180 \times 60 \times 60 \div \pi = 206264''\cdot 8, \text{ or } 57^\circ\ 17'\ 44''\cdot 8.$$

Here π denotes as usual the ratio of the circumference to the diameter, so that $2\pi a$ is the circumference of a circle of radius a; thus the c.m. (circular measure) of a right angle is $\frac{1}{2}\pi$ (radians), of the angle of an equilateral triangle is $\frac{1}{3}\pi$, and so on; and $\pi = 3{\cdot}14159265\ldots$, an *incommensurable* number.

By Euclid VI. 33, the area of the sector OAP is to that of the whole circle as the arc AP is to the circumference, or as θ to 2π; and the area of the circle being πa^2, the area of the sector is $\frac{1}{2}a^2\theta$, or one half the rectangle of the radius and the arc.

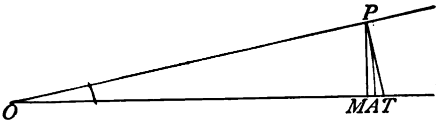


Fig. 5

In fig. 5 draw PM the perpendicular from P on OA, and let the tangent at P meet OA produced in T; then according to the definitions of Trigonometry,

$$\frac{\text{arc } AP}{OP} = \theta,\ \frac{OM}{OP} = \cos\theta,\ \frac{MP}{OP} = \sin\theta,$$

$$\frac{PT}{OP} = \tan\theta,\ \frac{OT}{OP} = \sec\theta,\ \frac{AM}{AO} = \text{vers}\,\theta;$$

also if TP produced meets OB drawn perpendicular to OA in V,

$$\frac{PV}{OP} = \cot\theta,\ \frac{OV}{OP} = \text{cosec}\,\theta;$$

we shall call these functions of θ the *circular functions.*

Then the triangle $OAP = \frac{1}{2}a^2 \sin\theta$, the sector $OAP = \frac{1}{2}a^2\theta$, and the triangle $OPT = \frac{1}{2}a^2\tan\theta$; and these are obviously in ascending order of magnitude, so that

$$\sin\theta,\ \theta,\ \tan\theta,$$

are also in ascending order, when θ is the c.m. of an angle less than a right angle, that is when $\theta < \frac{1}{2}\pi$.

Now make θ indefinitely small; then since, when $\theta = 0$,

$$\text{lt}\sin\theta/\tan\theta = \text{lt}\cos\theta = 1\,;$$

therefore also, $\text{lt}(\sin\theta)/\theta = 1$, and $\text{lt}(\tan\theta)/\theta = 1$; when $\theta = 0$; as assumed in the last article.

More generally, when $\theta = 0$,

$$\frac{\sin m\theta}{n\theta} = \frac{\sin m\theta}{m\theta}\frac{m}{n} = \frac{m}{n},\quad \frac{\tan m\theta}{n\theta} = \frac{m}{n};$$

thus, when $h = 0$, $\dfrac{\sin h^\circ}{h^\circ} = \dfrac{\tan h^\circ}{h^\circ} = \dfrac{\pi}{180}$,

$$\frac{\sin h'}{h'} = \frac{\tan h'}{h'} = \frac{\pi}{180\times 60},\ \frac{\sin h''}{h''} = \frac{\tan h''}{h''} = \frac{\pi}{180\times 60\times 60}.$$

Again, when $\theta = 0$,

$$\text{lt}\frac{\text{vers}\,\theta}{\theta} = \text{lt}\frac{AM}{\text{arc}\,AP} = \text{lt}\frac{AM}{AP}\text{lt}\frac{\text{chord}\,AP}{\text{arc}\,AP} = 0,$$

because $\text{lt}\dfrac{\text{chord}\,AP}{\text{arc}\,AP} = 1$, and $\text{lt}\dfrac{AM}{AP} = \text{lt}\sin\frac{1}{2}\theta = 0$.

As a numerical verification, the student may calculate from the Tables the values of $(\sin\theta)/\theta$, $(\tan\theta)/\theta$, and $(\text{vers}\,\theta)/\theta$, when θ is the c.m. of an angle of 3°, 2°, 1°, 30′, 10′, 1′, 30″, 10″, 1″.

When $\theta = \pi/n$, the c.m. of $180/n$ degrees, then $2MP$ and $2PT$ are sides of regular polygons of n sides, inscribed and circumscribed to the circle; so that $2na\sin\pi/n$ and $2na\tan\pi/n$ are the perimeters of the inscribed and circumscribed polygons of n sides; and the perimeter of the circle $2\pi a$ being intermediate in length, we can, by sufficiently increasing n, determine closer and closer limits between which π lies; in this manner Archimedes found that π lies between $3\frac{10}{70}$ and $3\frac{10}{71}$.

As a numerical exercise the student may calculate limits between which π lies, by taking $n=180$, 180×60, ... in $n\sin\theta/n$ and $n\tan\theta/n$.

17. *Differentiation of the Circular Functions.*

Having thus justified the differentiation of $\sin x$,

$$\frac{d\sin x}{dx}=\cos x,$$

where x is the circular measure of a variable angle, we proceed similarly with the other functions:—

$$\frac{d\cos x}{dx}=\operatorname{lt}\frac{\cos(x+h)-\cos x}{h}=\operatorname{lt}\frac{-2\sin(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h}$$

$$=-\operatorname{lt}\sin(x+\tfrac{1}{2}h)\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}=-\sin x.$$

$$\frac{d\tan x}{dx}=\operatorname{lt}\frac{\tan(x+h)-\tan x}{h}$$

$$=\operatorname{lt}\frac{\sin h}{h}\sec(x+h)\sec x=\sec^2 x.$$

$$\frac{d\cot x}{dx}=\operatorname{lt}\frac{\cot(x+h)-\cot x}{h}$$

$$=-\operatorname{lt}\frac{\sin h}{h}\operatorname{cosec}(x+h)\operatorname{cosec}x=-\operatorname{cosec}^2 x.$$

$$\frac{d\sec x}{dx}=\operatorname{lt}\frac{\sec(x+h)-\sec x}{h}=\operatorname{lt}\frac{\cos x-\cos(x+h)}{h\cos(x+h)\cos x},$$

$$=\sin x/\cos^2 x=\sec x\tan x.$$

Similarly

$$\frac{d\operatorname{cosec}x}{dx}=-\operatorname{cosec}x\cot x,\quad\frac{d\operatorname{vers}x}{dx}=\sin x.$$

The formulas of Trigonometry we have assumed in these differentiations are, with the usual notation,

$$\begin{aligned}\sin C-\sin D&=2\cos\tfrac{1}{2}(C+D)\sin\tfrac{1}{2}(C-D),\\ \cos C-\cos D&=-2\sin\tfrac{1}{2}(C+D)\sin\tfrac{1}{2}(C-D),\\ \tan A-\tan B&=\sin(A-B)\sec A\sec B.\end{aligned}$$

But these formulas are proved immediately by means of fig. 6, if we denote the angle AOP by C and AOQ by D; then $AOR=\frac{1}{2}(C+D)$, and $ROP=ROQ=\frac{1}{2}(C-D)$;

$$\begin{aligned}\sin C-\sin D&=\frac{LP}{OP}-\frac{MQ}{OQ}=2\frac{pP}{OP}=2\frac{pP}{RP}\frac{RP}{OP}\\ &=2\cos\tfrac{1}{2}(C+D)\sin\tfrac{1}{2}(C-D);\\ \cos C-\cos D&=\frac{OL}{OP}-\frac{OM}{OQ}=-2\frac{pR}{OP}=-2\frac{pR}{RP}\frac{RP}{OP}\\ &=-2\sin\tfrac{1}{2}(C+D)\sin\tfrac{1}{2}(C-D);\end{aligned}$$

and denoting AOR by A and ROP or ROQ by B,

$$\begin{aligned}\tan A-\tan B&=\frac{RT}{OR}-\frac{RQ}{OR}=\frac{QT}{OR}=\frac{MQ}{OQ}\frac{QT}{MQ}\frac{OQ}{OR}\\ &=\sin(A-B)\sec A\sec B.\end{aligned}$$

18. *Geometrical Differentiation of the Circular Functions.*

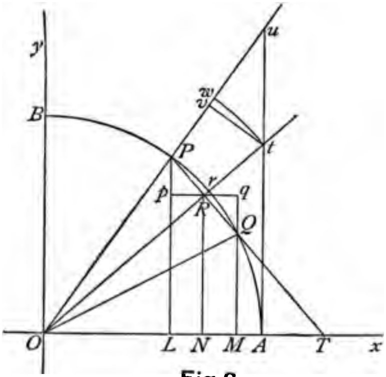


Fig.6

By means of fig. 6 we can illustrate geometrically the differentiation of the circular functions.

Thus if x is the c.m. of AOR, and h of the small angles POR, QOR; then employing the definition of § 5,

$$\frac{d\sin x}{dx}=\mathrm{lt}\frac{\sin(x+h)-\sin(x-h)}{2h}=\mathrm{lt}\frac{LP-MQ}{\mathrm{arc}\,PQ}$$

$$=\mathrm{lt}\frac{2pP}{\mathrm{arc}\,PQ}=\mathrm{lt}\frac{pP}{PR}\cdot\frac{\mathrm{chord}\,PQ}{\mathrm{arc}\,PQ}=\cos x.$$

$$\frac{d\cos x}{dx}=\mathrm{lt}\frac{OL-OM}{\mathrm{arc}\,PQ}$$

$$=-\mathrm{lt}\frac{2pR}{\mathrm{arc}\,PQ}=-\mathrm{lt}\frac{pR}{PR}\,\frac{\mathrm{chord}\,PQ}{\mathrm{arc}\,PQ}=-\sin x.$$

With the definition of § 3,

$$\frac{d\tan x}{dx}=\mathrm{lt}\frac{\tan(x+h)-\tan x}{h}=\mathrm{lt}\frac{tu}{\mathrm{arc}\,rP}$$

$$=\mathrm{lt}\frac{tu}{tv}\,\frac{tv}{\mathrm{arc}\,tw}\,\frac{\mathrm{arc}\,tw}{\mathrm{arc}\,rP}=\mathrm{lt}\frac{Ou}{OA}\,\frac{Ot}{OA}=\sec^2 x;$$

since $\dfrac{tu}{tv}=\dfrac{Ou}{OA}$, $\mathrm{lt}\dfrac{tv}{\mathrm{arc}\,tw}=1$ (§ 16), and $\dfrac{\mathrm{arc}\,tw}{\mathrm{arc}\,rP}=\dfrac{Ot}{OA}$.

$$\frac{d\sec x}{dx}=\mathrm{lt}\frac{wu}{\mathrm{arc}\,rP}=\mathrm{lt}\frac{wu}{vu}\,\frac{vu}{tv}\,\frac{tv}{\mathrm{arc}\,tw}\,\frac{\mathrm{arc}\,tw}{\mathrm{arc}\,rP}$$

$=\tan x\sec x$; since

$$\mathrm{lt}\frac{wu}{vu}=1,\ \mathrm{lt}\frac{vu}{tv}=\mathrm{lt}\frac{Au}{OA}=\tan x,\ \mathrm{lt}\frac{tv}{\mathrm{arc}\,tw}=1,\ \frac{\mathrm{arc}\,tw}{\mathrm{arc}\,rP}=\sec x.$$

19. By means of the formula of § 11 for the differentiation of a function of a function, we can differentiate any power of the circular functions of x, and the circular functions of any function of x; thus

$d(\sin x)^m/dx=m(\sin x)^{m-1}\cos x$; $d\sin y/dx=\cos y\,dy/dx$;

$$d(\sin y)^m/dx=m(\sin y)^{m-1}\cos y\,dy/dx.$$

As an exercise, the student may in the same manner differentiate with respect to x the functions $(\cos x)^n$, $(\tan x)^n$, $(\cot x)^n$, $(\sec x)^n$, $(\mathrm{cosec}\,x)^n$, $(\mathrm{vers}\,x)^n$, $(\mathrm{f}\,x)^n$, $\sin \mathrm{f}x$, $\{\mathrm{f}(\sin x)\}^n$, $(\cos \mathrm{f}x)^n$,...

Examples.—(1) Prove the following differentiations:

(i.) $y=\frac{1}{2}x+\frac{1}{4}\sin 2x$, $dy/dx=\cos^2 x$.
(ii.) $y=\frac{1}{2}x-\frac{1}{4}\sin 2x$, $dy/dx=\sin^2 x$.
(iii.) $y=\frac{1}{3}\cos^3 x-\cos x$, $dy/dx=\sin^3 x$.
(iv.) $y=\sin x-\frac{1}{3}\sin^3 x$, $dy/dx=\cos^3 x$.
(v.) $y=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x$, $dy/dx=\cos^4 x$,
and write down y when $dy/dx=\sin^4 x$.
(vi.) $y=\tan x-x$, $dy/dx=\tan^2 x$.
(vii.) $y=-\cot x-x$, $dy/dx=\cot^2 x$.
(viii.) $y=\frac{1}{3}\tan^3 x-\tan x+x$, $dy/dx=\tan^4 x$.
(ix.) $y=\frac{1}{3}\tan^3 x+\tan x$, $dy/dx=\sec^4 x$.

(2) From the definition of §3, differentiate with respect to x
$\sin 2x$, $\cos x/a$, $\tan(mx+n)$, $x\tan x$, $\tan x^2$, $\sin x^n$

(3) Differentiate x^n, $\sin x$, $\tan x^n$, with respect to x^m.

20. It will be useful also to draw the graphs of the circular functions; thus the equations

$$y/b=\pm\sin 2\pi x/a \text{ and } \pm\cos 2\pi x/a$$

represent in fig. 7, for two values of b, the plan of the spirals of a double-threaded screw of pitch a.

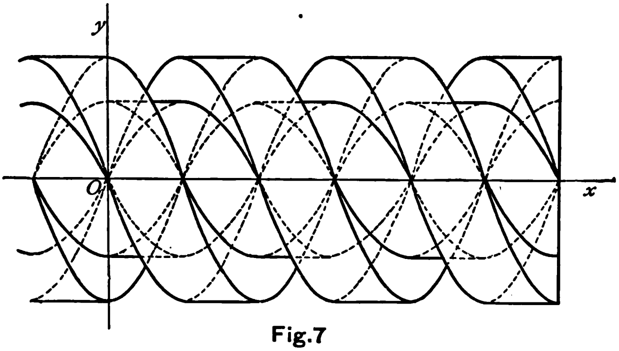


Fig.7

Examples.—Draw the curves: (1) $y=\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$, $\operatorname{vers} x$ (on one diagram).

(2) $y=\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$, $\operatorname{vers}^{-1}x$ (on one diagram); and superpose them on the preceding diagram.

(3) $y=x\sin x$, $(\sin x)/x$, $(\sin x)^2$, $\sin x^2$, $\sin\sqrt{x}$, $x\cot\frac{1}{2}\pi x$.

(4) $\sin x=\sin y$, $\cos x=\cos y$, $\tan x=\tan y$,
$(\sin x)^2+(\sin y)^2=1$, $(\tan x)^2+(\tan y)^2=1$.

(5) Prove that in the ellipse $(x/a)^2+(y/b)^2=1$, we may put $x=a\cos\theta$, $y=b\sin\theta$ (θ is then called the *excentric angle*); and $AOp=\theta$, and the equations of the tangent and normal are (fig. 8)

$$(x/a)\cos\theta+(y/b)\sin\theta=1;\ ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2.$$

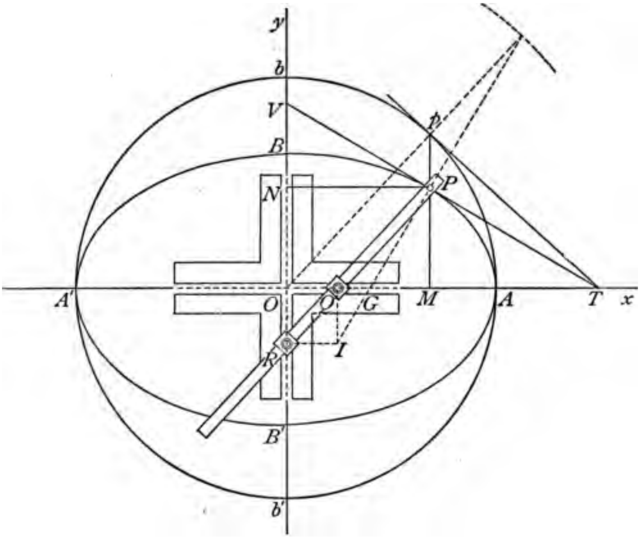


Fig.8

(Fig. 8 shows the *elliptic trammel* or *compasses*, the theory of which is obvious from this example.)

If $z=c$, a constant, then V is the volume of a right cylinder on the base A, so that

$$V=c\iint dxdy, \text{ and } A=\iint dxdy;$$

and the area A may be considered as built up of the infinitesimal elements of area $dxdy$, the limits of integration being taken so as to include all the elements of area in A.

135. If the surface $z=\mathrm{f}(x, y)$ becomes a plane, then

$$z=lx+my+n,$$

and $V=\iint(lx+my+n)dxdy=(l\bar{x}+m\bar{y}+n)A=\bar{z}A$, where $\bar{z}$ is the ordinate of the plane standing on the centroid of the base A; this is the formula for the volume cut off a cylinder by a slant plane and a cross section, or by two slant planes.

By means of this principle we can calculate the volume of an earthwork dam across a valley, of which the contour lines are assumed to be parallel straight lines; also the volume of a groin formed by the intersection of two equal barrel vaults, crossing at right angles.

136. With polar (cylindrical) coordinates r, θ, and z, and double integration, the infinitesimal element of area enclosing a point (r, θ) in the plane of $z=0$ may be supposed bounded by circles of radii $r-\frac{1}{2}dr$ and $r+\frac{1}{2}dr$, and vectors from O making angles $\theta-\frac{1}{2}d\theta$ and $\theta+\frac{1}{2}d\theta$ with Ox; so that its area is

$$\tfrac{1}{2}\{(r+\tfrac{1}{2}dr)^2-(r-\tfrac{1}{2}dr)^2\}d\theta=dr\,.\,rd\theta;$$

and
$$A=\iint rdrd\theta,\quad V=\iint zrdrd\theta;$$

the limits of r and θ being taken so as to include all the elements in the area A.

In the general change from the independent variables x, y to any two new variables u, v, the element of area bounded by the curves u, v, $u+du$, $v+dv$ is ultimately a parallelogram, the coordinates of whose angular points are, relatively to the corner (x, y),

$$\frac{\partial x}{\partial u}du,\ \frac{\partial y}{\partial u}du;\ \frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv,\ \frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv;\ \frac{\partial x}{\partial v}dv,\ \frac{\partial y}{\partial v}dx;$$

and therefore the area of this elementary parallelogram is

$$\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)dudv=\frac{\partial(x, y)}{\partial(u, v)}dudv;$$

so that now, with z expressed as a function of u, v,

$$A=\iint\frac{\partial(x, y)}{\partial(u, v)}dudv,\quad V=\iint z\frac{\partial(x, y)}{\partial(u, v)}dudv.$$

Thus, for instance, in Thermodynamics, the pressure p and volume v of unit mass of a perfect gas are connected with θ, the absolute temperature, and ϕ, the *entropy*, by the relations (Maxwell, *Heat*)

$$pv=R\theta=(m-n)\theta,\ p^nv^m=Ce^\phi;$$

and, taking θ, ϕ as independent variables,

$$\frac{1}{p}\frac{\partial p}{\partial\theta}+\frac{1}{v}\frac{\partial v}{\partial\theta}=\frac{1}{\theta},\quad \frac{1}{p}\frac{\partial p}{\partial\phi}+\frac{1}{v}\frac{\partial v}{\partial\phi}=0;$$

$$\frac{n}{p}\frac{\partial p}{\partial\theta}+\frac{m}{v}\frac{\partial v}{\partial\theta}=0,\quad \frac{n}{p}\frac{\partial p}{\partial\phi}+\frac{m}{v}\frac{\partial v}{\partial\phi}=1;$$

so that,

$$\frac{m-n}{p}\frac{\partial p}{\partial\theta}=\frac{m}{\theta},\quad \frac{m-n}{p}\frac{\partial p}{\partial\phi}=-1;$$

$$\frac{m-n}{v}\frac{\partial v}{\partial\theta}=-\frac{n}{\theta},\quad \frac{m-n}{v}\frac{\partial v}{\partial\phi}=\ 1;$$

and

$$\frac{\partial(p, v)}{\partial(\theta, \phi)}=\frac{1}{m-n}\frac{pv}{c\theta}=1.$$

Therefore the area on the p, v diagram of the *Carnot cycle* bounded by the two *isothermals* θ_1, θ_2 and by the two *adiabatics* ϕ_1, ϕ_2 is $(\theta_1-\theta_2)(\phi_1-\phi_2)$.

137. *The Sign to be attributed to an Area.*

Suppose a point to travel once round the closed ov area A, an indicator diagram for instance, so as always have the interior of the curve on the *left* hand (fig. 44).

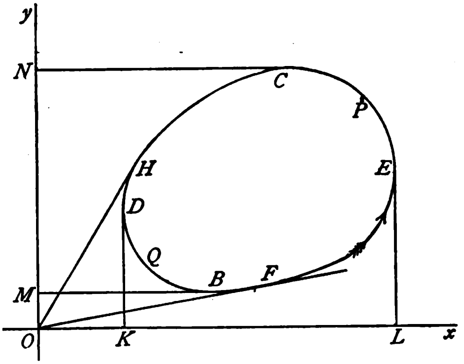


Fig. 44

Then $$A = \iint dx dy = \int x dy = \int x \frac{dy}{dt} dt,$$

taken round the perimeter of the curve.

From B to C along BPC, dy/dt is positive,

and $$\int x \frac{dy}{dt} dt = \text{ area } MBPCN;$$

and from C to B along CQB, dy/dt is negative,

and $$\int x \frac{dy}{dt} dt = -\text{area } MBQCN;$$

so that, taken round the curve,

$$\int x \frac{dy}{dt} dt, \text{ or } \int x dy = A, \text{ the area of the closed curve.}$$

But $$\iint dy dx = \int y dx = \int y \frac{dx}{dt} dt;$$

and from E to D along EPD, dx/dt is negative, so that

$$\int y\frac{dx}{dt}dt = -\text{area } LEPDK;$$

and from D to E along DQE, dx/dt is positive, so that

$$\int y\frac{dx}{dt}dt = \quad \text{area } LEQDK;$$

and therefore, taken round the curve,

$$\int y\frac{dx}{dt}dt, \text{ or } \int ydx = -A;$$

Therefore, taken round the curve,

$$\int\left(y\frac{dx}{dt}+x\frac{dy}{dt}\right)dt, \text{ or } \int(ydx+xdy)=0;$$

and $$ydx+xdy=d(xy)$$

is called a *perfect differential;* its integral between two limits is independent of the intermediate values of x and y and of the path described between the limits; so that, taken round any closed path, the integral is zero.

But $$A=\int\tfrac{1}{2}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt;$$

and $xdy-ydx$ is *not* a perfect differential, so that its integrated value depends on the path taken between two limits; and in a closed path it represents twice the area enclosed by the path.

Similar theorems hold for the integrals

$$\iint zdxdy \text{ and } \iint zdydx.$$

138. Changing to polar coordinates by putting

$$x=r\cos\theta,\ y=r\sin\theta,$$

then $$x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{d\theta}{dt};$$

so that $$A=\int\tfrac{1}{2}r^2\frac{d\theta}{dt}dt=\int\tfrac{1}{2}r^2d\theta,$$

taken round the curve.

For an origin inside a closed oval curve, the limits of θ may be taken as O and 2π.

But if the origin O is outside the area, draw the tangents OF, OH to the curve from O; then along FPH, $d\theta/dt$ is positive, and

$$\int \tfrac{1}{2}r^2\frac{d\theta}{dt}dt = \text{ area } OFPCHO;$$

but along HRF, $d\theta/dt$ is negative, and

$$\int \tfrac{1}{2}r^2\frac{d\theta}{dt}dt = -\text{area } OFQHO;$$

so that, taken round the curve,

$$A = \int \tfrac{1}{2}r^2\frac{d\theta}{dt}dt = \int \tfrac{1}{2}r^2 d\theta = \int \tfrac{1}{2}x^2 dv, \text{ if } y/x = v = \tan\theta.$$

Although not a perfect differential, $xdy - ydx$ can be made one by dividing by x^2+y^2, and then

$$\frac{xdy-ydx}{x^2+y^2} = d\tan^{-1}\frac{y}{x} = d\theta;$$

so that $$\int\frac{xdy-ydx}{x^2+y^2} = \int d\theta = 2\pi, \text{ or } O,$$

according as the origin O is inside or outside the curve.

Consequently if each point of the contour is displaced through small distances c^2x/r^2, c^2y/r^2, parallel to the axes, the change in the area will be $2\pi c^2$ or zero, according as the origin is inside or outside the contour.

139. A convenient independent variable to take is s, the length of the arc of the perimeter measured from a fixed point; so that the point P may be supposed to move with unit velocity; and now, with p denoting the length of the perpendicular from O on the tangent at P (§ 9),

$$x\frac{dy}{ds} - y\frac{dx}{ds} = p;$$

so that
$$A=\int\tfrac{1}{2}pds=\int\tfrac{1}{2}p\frac{ds}{dr}dr$$
$$=\int\tfrac{1}{2}p\sec\phi dr=\int\tfrac{1}{2}prdr/\surd(r^2-p^2).$$

When fig. 44 represents an indicator diagram, and KL the reduced stroke of the piston, while the ordinate y represents the pressure of the steam, the pencil will describe the contour with the area to the left, when the steam pressure is urging the piston from L to K.

The diagram taken on the return stroke from the other end of the cylinder will be described in the opposite sense, with the area on the right hand of the describing pencil.

Sometimes a loop is found on the diagram, described in the reverse sense; this loop shows that a cushioning effect takes place, which requires attention, as the negative area of this loop represents so much lost work.

In the general case, where the perimeter cuts itself a number of times, then the area obtained by integrating once round a loop will be positive or negative with the above formulas, according as the area is on the left or right of the describing point, as it travels round the curve. (Clifford, *Common Sense of the Exact Sciences;* Cremona, *Graphical Statics.*)

Familiar instances of such looped contours are seen in Lissajous's figures (Ganot, *Physics*, § 281), whose general equation may be written as (§ 103)
$$m\sin^{-1}x/a-n\sin^{-1}y/b=\text{a constant},$$
or
$$x/a=\sin(nt+\epsilon),\ y/b=\sin(mt+\epsilon').$$

Also with the polar curves
$$r=a\cos(m\theta/n),$$
where m and n are integers, curves seen on the back of an engine-turned watch.

140. *The Planimeter.*

This instrument in the most usual form, that invented by Amsler of Schaffhausen, consists of two bars OA, AP, pivoted at O and jointed at A, and carrying in PA produced a small graduated roller R, with axis fixed parallel to PA (fig. 45).

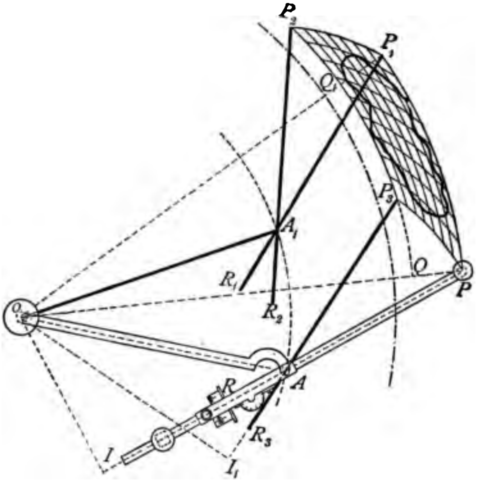


Fig. 45

The instrument is used to measure areas; the pointer P is carried round the perimeter of the curve whose area is required, an indicator diagram, or the section of a ship for instance; the roller R, which rolls and slides on the plane of the paper, then registers the area.

If precision is required, the point P may be carried say ten times round the perimeter, and the reading of the roller be divided by ten.

To explain the theory of the instrument, we shall suppose the pointer P to travel round a finite area $PP_1P_2P_3$ (something like a Carnot cycle in Thermodynamics), in which PP_1, P_2P_3 are circular arcs described round O as centre, and P_1P_2, P_3P, are arcs round A_1, A as centres; in this way we analyse the motion due to the joints O and A separately.

Let $OA=a$, $AP=b$, $AR=c$; and let the direction of a positive motion of the roller, as marked by the graduations, be that which on a right-handed screw would give a motion in the direction AR.

Drop the perpendicular OI from O on AR.

(i.) Fix the joint A, and move P to P_1 by rotation round O on the circle PP_1 through an angle θ; the angular velocity $d\theta/dt$ will give to the roller R the component velocities $OI.d\theta/dt$ in the direction IR and $IR.d\theta/dt$ perpendicular to IR; the first component drags the roller over the paper, and the second component makes the roller turn with circumferential velocity $IR.d\theta/dt$; and therefore the whole travel of the roller, or its graduations, will be $IR.\theta$.

(ii.) Fix the joint O, and move P_1 to P_2 by rotation round the joint A_1 through an angle ϕ; the angular velocity $d\phi/dt$ of AP will communicate a circumferential velocity $-cd\phi/dt$ to the roller; and the travel of the roller will therefore be $c\phi$, backwards.

(iii.) Fix the joint A, and move P_2 to P_3 by rotation round O through an angle θ; the roller will now move backwards, and its travel will be $I_1R.\theta$.

(iv.) Fix the joint O, and move P_3 to P by rotation round A through an angle ϕ; the travel of the roller will be $c\phi$, forwards; this cancels the travel of (ii.).

In completing the finite circuit $PP_1P_2P_3$, the total forward travel of the roller will then be $(IR-I_1R)\theta$.

But the area $PP_1P_2P_3 =$ area PP_1Q_1Q

$$= \text{sector } OPP_1 - \text{sector } OQQ_1$$

$$= \tfrac{1}{2}(OP^2 - OP_3^2)\theta$$

$$= \tfrac{1}{2}(OA^2 + AP^2 + 2AI.AP - OA^2 - AP^2 - 2AI_1.AP)\theta$$

$$= b(AI - AI_1)\theta = b(IR - I_1R)\theta$$

$= b$ times the travel of the roller.

Thus by altering the length of b by an adjustment of the instrument, which allows the arm AP to slide in the sleeve AR and be clamped, the area can be read off in any required unit, say the square inch or square centimetre.

Any irregular area, such as for instance an indicator diagram or the cross section of a ship, must be supposed built up of infinitesimal elements formed in the same manner as $PP_1P_2P_3$; and will be read off when the pointer P completes a circuit of the perimeter, both joints being now free to turn simultaneously.

When I coincides with R, the roller will not turn, and then P describes a circle called the zero circle, represented by the middle dotted circular line (fig. 45) of radius

$$\surd(OR^2+RP^2) = \surd\{a^2 - c^2 + (b+c)^2\} = \surd(a^2+b^2+2bc).$$

When I lies on the same side of R as A, the travel of the roller is reversed, but in a complete circuit the reading is unaltered.

If however the origin O is taken inside the area to be measured, the area of the zero circle must be added to the reading of the roller.

Prof. G. B. Mathews explains the theory of the Planimeter in a slightly different manner, by (i.) moving OA to OA_1, and P to P_1, keeping AP parallel to itself; and then the curvilinear area $APP_1A_1=b$ times the travel of the roller;

(ii.) moving A_1P_1 to A_1P_2 by rotation round A_1;

(iii.) moving OA_1 to OA and P_2 to P_3, keeping A_1P_2 parallel to itself; and then the curvilinear area $AA_1P_2P_3=b$ times the travel of the roller, backwards;

(iv.) completing the circuit by moving AP_3 to AP by rotation round A, when the sector $APP_3=$ sector $A_1P_1P_2$, and the travel of the roller cancels the travel in step (ii.).

Therefore the

$$\text{area } PP_1P_2P_3=\text{area } APP_1A_1-\text{area } A_1P_2P_3A$$
$$=b \text{ times the travel of the roller.}$$

The end A may be guided in a slot of any form and the area will be read off as before; a straight slot is often employed, with the advantage that the pencil P can then cover a greater area; and with appropriate mechanism can be made to register the moment of the area, and its moment of inertia about the straight line of the slot.

(J. F. Bramwell, *British Association*, 1872;
H. S. Hele Shaw, *Proc. I.C.E.*, 1885.)

Examples.

(1) Prove that the area is $\pi(Bc-bC)$ of the curve, an ellipse (Ex. 4, p. 203), given by

$$x=a+b\cos\theta+c\sin\theta,\ y=A+B\cos\theta+C\sin\theta.$$

(2) Trace and find the area of the curve

$$(r-a\cos\theta)^2=a^2\cos 2\theta.$$

(3) Prove that the volume cut off from the surface

$$z^n=Ax^2+2Hxy+By^2$$

by the plane $z=c$ is $n/(n+1)$ of the cylinder on the same base.

141. *Functions of three or more Independent Variables.* A function of three independent variables, x, y, z, denoted by $f(x, y, z)$, may be supposed to represent some function of the position of a point in space whose coordinates are x, y, z; for instance, the density or temperature or pressure at the point.

Then $f(x, y, z) = C$, a constant, would imply a relation connecting x, y, z, and would be the equation of a surface; for instance, a surface of equal density, or pressure.

If V denotes the volume contained in a closed surface S, then

$$V = \iiint dxdydz,$$

the integration including all the infinitesimal brick shaped elements of volume $dxdydz$ (fig. 43) contained in S.

When the *density* ρ within the surface S is variable and a given function of x, y, z, then the *mass* M contained by the surface S is given by the triple integration

$$M = \iiint \rho dxdydz, \text{ so that } \frac{\partial^3 M}{\partial x \partial y \partial z} = \rho,$$

and the mass is the *space integral* of the density ρ throughout the volume V; while $\bar{x}, \bar{y}, \bar{z}$, the coordinates of the centre of mass, and k_x, k_y, k_z, the radii of gyration about Ox, Oy, Oz, are given by

$$\bar{x}M = \iiint x\rho dxdydz, \qquad \bar{y}M = \iiint y\rho dxdydz,$$
$$\bar{z}M = \iiint z\rho dxdydz;$$
$$Mk_x^2 = \iiint (y^2 + z^2)\rho dxdydz, \quad Mk_y^2 = \iiint (z^2 + x^2)\rho dxdydz,$$
$$Mk_z^2 = \iiint (x^2 + y^2)\rho dxdydz.$$

Functions of more than three independent variables cannot be interpreted geometrically without the introduction of the fiction of space of more than three dimensions, a thing which is inconceivable.

A function $F(t, x, y, z)$ of four independent variables t, x, y, z, may however be interpreted, as in Hydrodynamics, as representing the velocity, or density, or pressure, at the time t at a point in space whose coordinates are x, y, z.

142. We have used the words *mass* and *density;* the *mass* of a body is the quantity measured by the balance against certain standard lumps of metal, called *weights* in the Acts of Parliament (French, *poids*, German, *Gewichte*), the standard in this country being the Pound Weight, and in the Metric System the Kilogramme of 1000 grammes.

The *density* of a body is defined as the number of units of mass in the unit of volume; with British units, the density is the number of lbs. in a cubic foot of the substance, and with Metric units is the number of grammes in a cubic centimetre, or of *tonnes* of 1000 kilogrammes per cubic metre.

The units of length are thus the *foot* in the British System, and the *metre* or *centimetre* in the Metric System; while the unit of time in universal use for theoretical investigations is the *second*, the mean solar sexagesimal second.

When therefore we speak of a time t, we mean t seconds; and coordinates x, y, z are measured in the unit of length, which is either the *foot* or else the *metre* or *centimetre.*

For practical purposes there are only three systems of fundamental units which need be considered,

(i.) the British foot-pound-second (F.P.S.) system;
(ii.) the C.G.S. (centimetre-gramme-second) system;
(iii.) the metre-kilogramme-second (M.K.S.) system;

and from these fundamental units of length, mass, and time, all other units, of *area, volume, density, velocity, acceleration, momentum, energy, force*, etc., may be derived.

*143. *Spherical Polar Coordinates.*

In this system of coordinates the position of a point P on a sphere with centre at O is defined by θ, its angular distance from a fixed pole N on the sphere, and by ϕ the angle which the plane ONP makes with a fixed initial plane; so that on the terrestrial sphere, ϕ will be the longitude and $\frac{1}{2}\pi-\theta$ the north latitude, if N denotes the north pole.

By taking r the radius of the sphere as variable, we can define the position of any point in space by means of the three quantities r, θ, ϕ, called the spherical polar coordinates in space.

With ON coincident with Oz, and with the plane xOz as the prime meridian, these coordinates are connected with the orthogonal coordinates x, y, z, by means of the relations $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$.

If we replace $r\sin\theta$ by ϖ, then ϖ, ϕ, z, are the coordinates in the cylindrical system (§ 132).

The element of volume cut out by the spheres $r-\frac{1}{2}dr$ and $r+\frac{1}{2}dr$, by the cones $\theta-\frac{1}{2}d\theta$ and $\theta+\frac{1}{2}d\theta$, and by the planes $\phi-\frac{1}{2}d\phi$ and $\phi+\frac{1}{2}d\phi$, will be ultimately

$$dr\,.\,rd\theta\,.\,r\sin\theta d\phi;$$

so that by triple integration

$$V=\iiint r^2\sin\theta drd\theta d\phi,\text{ and }\frac{\partial^3 V}{\partial r\partial\theta\partial\phi}=r^2\sin\theta;$$

while the mass M and its centre of mass, for variable density ρ, are given by

$$\begin{aligned}M&=\iiint\rho r^2\sin\theta drd\theta d\phi,\\ \bar{x}M&=\iiint\rho r^3\sin^2\theta\cos\phi drd\theta d\phi,\\ \bar{y}M&=\iiint\rho r^3\sin^2\theta\sin\phi drd\theta d\phi,\\ \bar{z}M&=\iiint\rho r^3\sin\theta\cos\theta drd\theta d\phi.\end{aligned}$$

*144. *Space, Surface, and Line Integrals.*

Consider a fixed closed surface S, and a function X of the coordinates x, y, z of a point in space.

Then in the triple integration extending throughout the volume enclosed by the surface S, called a *space integral,*

$$\iiint \frac{\partial X}{\partial x} dxdydz = \iint (-X_1 + X_2 - X_3 + \ldots) dydz,$$

where $X_1, X_2, X_3, \ldots$ denote the values of X where a point moving from $-\infty$ to ∞ parallel to the axis Ox successively enters and leaves the interior of the surface S.

Denoting by $l_1, l_2, l_3, \ldots$ the cosines of the angles which the *outward* drawn normals of the surface S at these points make with Ox, then

$$dydz = -l_1 dS_1 = l_2 dS_2 = -l_3 dS_3 = \ldots,$$

supposing the infinitesimal prism on the base $dydz$ parallel to Ox to cut out the elements of surface $dS_1, dS_2, dS_3, \ldots,$ on entering and leaving the surface S.

Therefore, denoting the element of volume by dV,

$$\iiint \frac{\partial X}{\partial x} dV = \iint (l_1 X_1 dS_1 + l_2 X_2 dS_2 + \ldots) = \iint lXdS \ldots \text{(i.)},$$

the double integration extending over the surface S, and this is called a *surface integral;* so that a volume integral can always be expressed as a surface integral.

Similarly, with Y, Z other given functions of x, y, z,

$$\iiint \frac{\partial Y}{\partial y} dV = \iint mYdS, \quad \iiint \frac{\partial Z}{\partial z} dV = \iint nZdS;$$

where m, n denote the cosines of the angles the outward drawn normal of the surface S makes with Oy, Oz.

Therefore, by addition

$$\iiint \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) dV = \iint (lX + mY + nZ) dS \ldots\ldots \text{(ii.)}.$$

Thus $V = \iiint dxdydz = \frac{1}{3} \iint (lx + my + nz) dS.$

For instance, integrating over the surface of the ellipsoid, p denoting the perpendicular from the centre upon the tangent plane,

$$\int pdS = 4\pi abc, \text{ and } \int dS/p = \tfrac{4}{3}\pi(bc/a + ca/b + ab/c);$$

and, as an exercise, the student may calculate $\int dS/p$ and $\int dS/p^3$ for the hyperboloids, as well as for the ellipsoid.

Again, suppose X, Y, Z are the component forces per unit of volume acting throughout a fluid at rest, in which the pressure at any point is represented by p; then the equation of equilibrium of the fluid within the closed surface S, obtained by resolving parallel to Ox, is

$$\iiint XdV = \iint lpdS.$$

But changing the surface integral into a space integral,

$$\iint lpdS = \iiint \frac{\partial p}{\partial x} dV,$$

so that $\quad \dfrac{\partial p}{\partial x} = X$; and similarly $\dfrac{\partial p}{\partial y} = Y$, $\dfrac{\partial p}{\partial z} = Z$;

or $\qquad dp = Xdx + Ydy + Zdz,$

so that the *space variation* of the pressure of a fluid at rest in any direction is equal to the component force per unit of volume in that direction; and surfaces of equal pressure are cut orthogonally by the lines of force.

As another illustration, we may suppose X, Y, Z to represent the components of flux (estimated with British units in lb. per square foot per second) of a fluid in motion; then $\iint(lX + mY + nZ)dS$ represents the number of lb. which is flowing out across the surface S per second; while if ρ denotes the density, in lb. per cubic foot, of the fluid at any point of the interior of S, so that the mass M within S is $\iiint \rho dV$ lb., then $\partial M/\partial t$ represents the rate of increase, in lb. per second, of the quantity of fluid inside S.

Equating this gain and loss,

$$\partial M/\partial t + \iint (lX + mY + nZ) dS = 0;$$

and, replacing by space integrals,

$$\iiint \left(\frac{\partial \rho}{\partial t} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) dV = 0,$$

leading to the *equation of continuity* in Hydrodynamics, when we replace X, Y, Z by ρu, ρv, ρw, where u, v, w denote the components of *velocity* of the fluid.

In a plane, dA denoting an element of the area A, of which ds represents an element of length of the closed contour s, equation (ii.) becomes

$$\iint \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) dA = \int (lX + mY) ds \ldots\ldots\ldots\ldots\ldots\text{(iii.)},$$

thus expressing a *surface* integral by a *line* integral; and $l = dy/ds$, $m = -dx/ds$;

or, if $F(x, y) = 0$ is the equation of the contour,

$$l = \frac{\partial F}{\partial x} \Big/ \sqrt{\left\{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2\right\}},$$

$$m = \frac{\partial F}{\partial y} \Big/ \sqrt{\left\{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2\right\}}.$$

A similar theorem connects the surface integral on a curved surface S, which is a portion of a closed surface, with a line integral round the edge of S;

$$\iint \left\{ l\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + m\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + n\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) \right\} dS$$

$$= \int \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) ds \ldots\ldots\ldots\ldots\ldots\text{(iv.)}$$

(Maxwell, *Electricity*, i., p. 25), this surface integral vanishing, by equation (ii.), for a closed surface.

For a plane surface, we may take $l=0, m=0, n=1$; and now, as in (iii.),

$$\iint\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)dA=\int\left(X\frac{dx}{ds}+Y\frac{dy}{ds}\right)ds.$$

*145. *Green's Theorem.*

Now suppose U and U' are given functions of x, y, z; then from equation (i.), integrating by parts,

$$\iiint\frac{\partial U}{\partial x}\frac{\partial U'}{\partial x}dV=\iint lU'\frac{\partial U}{\partial x}dS-\iiint U'\frac{\partial^2 U}{\partial x^2}dV\text{....(v.)};$$

and therefore, denoting by $-\nabla^2$ the operator

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},$$

$$\iiint\left(\frac{\partial U}{\partial x}\frac{\partial U'}{\partial x}+\frac{\partial U}{\partial y}\frac{\partial U'}{\partial y}+\frac{\partial U}{\partial z}\frac{\partial U'}{\partial z}\right)dV$$

$$=\iint U'\left(l\frac{\partial U}{\partial x}+m\frac{\partial U}{\partial y}+n\frac{\partial U}{\partial z}\right)dS+\iiint U'\nabla^2 U dV$$

$$=\iint U'\frac{\partial U}{\partial \nu}dS+\iiint U'\Delta^2 U dV,\text{ ....(vi.)},$$

and therefore, by symmetry,

$$=\iint U\frac{\partial U'}{\partial \nu}dS+\iiint U\Delta^2 U'dV,\text{ ....(vii.)},$$

where $$l\frac{\partial U}{\partial x}+m\frac{\partial U}{\partial y}+n\frac{\partial U}{\partial z}=\frac{\partial U}{\partial \nu},$$

representing the rate of growth of U in the direction of the outward drawn normal of S.

Equations (vi.) and (vii.) constitute *Green's Theorem*, a theorem of great use in the mathematical theories of Electricity and Magnetism.

(An *Essay on Electricity and Magnetism*, by G. Green; edited by N. M. Ferrers.)

*146. *Change of the Variables in Space Integrals.*

Generally in changing from x, y, z to any new independent variables u, v, w, we may consider that space is divided up into elements of volume bounded by the surfaces for which u, v, w are constant; and now the element of volume will be changed from $dxdydz$ to

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}dudvdw,$$

$\frac{\partial(x, y, z)}{\partial(u, v, w)}$ denoting the determinant $\begin{vmatrix} \frac{\partial x}{\partial u}, & \frac{\partial y}{\partial u}, & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v}, & \frac{\partial y}{\partial v}, & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w}, & \frac{\partial y}{\partial w}, & \frac{\partial z}{\partial w} \end{vmatrix}$

called the Jacobian of x, y, z with respect to u, v, w.

Denoting by ds, the element of length, then ds^2 becomes changed from $dx^2 + dy^2 + dz^2$ to

$$\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw\right)^2 + \left(\frac{\partial y}{\partial u}du + \ldots\right)^2 + \left(\frac{\partial z}{\partial u}du + \ldots\right)^2$$

$$= A^2du^2 + B^2dv^2 + C^2dw^2 + 2Ddvdw + 2Edwdu + 2Fdudv,$$

$$A^2 = \frac{\partial x^2}{\partial u^2} + \frac{\partial y^2}{\partial u^2} + \frac{\partial z^2}{\partial u^2}, \ldots, \quad D = \frac{\partial x}{\partial v}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial v}\frac{\partial y}{\partial w} + \frac{\partial z}{\partial v}\frac{\partial z}{\partial w}, \ldots.$$

It is convenient to choose the new variables u, v, w, so that the corresponding surfaces cut at right angles, and then D, E, F vanish; while

$$\frac{1}{A}\frac{\partial x}{\partial u}, \frac{1}{A}\frac{\partial y}{\partial u}, \frac{1}{A}\frac{\partial z}{\partial u}$$

represent the cosines of the angles the normal to the surface u makes with the axes Ox, Oy, Oz.

But with x, y, z as independent variables, and denoting $\frac{\partial u^2}{\partial x^2}+\frac{\partial u^2}{\partial y^2}+\frac{\partial u^2}{\partial z^2}$ by h_1^2, $\frac{\partial v^2}{\partial x^2}+\dots$ by h_2^2, $\frac{\partial w^2}{\partial x^2}+\dots$ by h_3^2;

then $$\frac{1}{h_1}\frac{\partial u}{\partial x},\ \frac{1}{h_1}\frac{\partial u}{\partial y},\ \frac{1}{h_1}\frac{\partial u}{\partial z}$$

are also the cosines of the angles which the normal to the surface u makes with Ox, Oy, Oz; so that

$$\frac{1}{h_1}\frac{\partial u}{\partial x}=\frac{1}{A}\frac{\partial x}{\partial u},\ \frac{1}{h_1}\frac{\partial u}{\partial y}=\frac{1}{A}\frac{\partial y}{\partial u},\ \frac{1}{h_1}\frac{\partial u}{\partial z}=\frac{1}{A}\frac{\partial z}{\partial u},$$

and similarly,

$$\frac{1}{h_2}\frac{\partial v}{\partial x}=\frac{1}{B}\frac{\partial x}{\partial v},\ \frac{1}{h_2}\frac{\partial v}{\partial y}=\frac{1}{B}\frac{\partial y}{\partial v},\ \frac{1}{h_2}\frac{\partial v}{\partial z}=\frac{1}{C}\frac{\partial z}{\partial v};$$

$$\frac{1}{h_3}\frac{\partial w}{\partial x}=\frac{1}{C}\frac{\partial x}{\partial w},\ \frac{1}{h_3}\frac{\partial w}{\partial y}=\frac{1}{C}\frac{\partial y}{\partial w},\ \frac{1}{h_3}\frac{\partial w}{\partial z}=\frac{1}{B}\frac{\partial z}{\partial w};$$

Also, since $$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz,$$

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz,$$

$$dw=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\frac{\partial w}{\partial z}dz;$$

therefore, multiplying these equations by

$$\frac{1}{h_1^2}\frac{\partial u}{\partial x},\ \frac{1}{h_2^2}\frac{\partial v}{\partial x},\ \frac{1}{h_3^2}\frac{\partial w}{\partial x},$$

and adding, $dx=\frac{1}{h_1^2}\frac{\partial u}{\partial x}du+\frac{1}{h_2^2}\frac{\partial v}{\partial x}dv+\frac{1}{h_3^2}\frac{\partial w}{\partial x}dw$;

so that $\frac{\partial x}{\partial u}=\frac{1}{h_1^2}\frac{\partial u}{\partial x}, \dots$; or $Ah_1=Bh_2=Ch_3=1$.

Now denoting by ds_1, ds_2, ds_3, the elements of the normals, intersections of the surfaces u, v, w,

$$ds_1=A\,du,\ ds_2=B\,dv,\ ds_3=C\,dw;$$

so that $\frac{\partial(x,y,z)}{\partial(u,v,w)}=ABC$, and $\frac{\partial(u,v,w)}{\partial(x,y,z)}=h_1h_2h_3=1/ABC$

Now if V denotes any function of x, y, z, or u, v, w, then

$$\frac{\partial V^2}{\partial x^2}+\frac{\partial V^2}{\partial y^2}+\frac{\partial V^2}{\partial z^2}=h_1{}^2\frac{\partial V^2}{\partial u^2}+h_2{}^2\frac{\partial V^2}{\partial v^2}+h_3{}^2\frac{\partial V^2}{\partial w^2};$$

also,

$$-\nabla^2 V=h_1h_2h_3\left\{\frac{\partial}{\partial u}\left(\frac{h_1}{h_2h_3}\frac{\partial V}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{h_2}{h_3h_1}\frac{\partial V}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{h_3}{h_1h_2}\frac{\partial V}{\partial w}\right)\right\};$$

which are readily proved by taking the axes parallel to the normals to u, v, w at the point.

With spherical polar coordinates r, θ, ϕ (§ 144),

$$ds^2=dx^2+dy^2+dz^2=dr^2+r^2d\theta^2+r^2\sin\theta d\phi^2,$$

so that $\quad A=1,\ B=r,\ C=r\sin\theta;$

$$-\nabla^2 V=\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}.$$

*147. *Confocal Quadrics.*

A familiar instance occurs with confocal quadric surfaces, where, with λ, μ, ν for new variables,

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1,\ \frac{x^2}{a^2+\mu}+\ldots=1,\ \frac{x^2}{a^2+\nu}+\ldots=1;$$

whence, by solution,

$$x^2=\frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)},\ y^2=\ldots,\ z^2=\ldots;$$

and $\quad A^2=\dfrac{(\lambda-\mu)(\lambda-\nu)}{4(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)},\ B^2=\ldots,\ C^2=\ldots.$

Denoting by p the perpendicular from the centre on the tangent plane of the surface λ, then we can show that

$$\frac{1}{p^2}=\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2},\ \frac{\partial\lambda}{\partial x}=\frac{2p^2x}{a^2+\lambda},\ \ldots,\ h_1{}^2=4p^2,$$

and $\quad \dfrac{\partial^2\lambda}{\partial x^2}+\dfrac{\partial^2\lambda}{\partial y^2}+\dfrac{\partial^2\lambda}{\partial z^2}=2p^2\left(\dfrac{1}{a^2+\lambda}+\dfrac{1}{b^2+\lambda}+\dfrac{1}{c^2+\lambda}\right);$

and thence that $V=\mathrm{f}\lambda$ will satisfy the condition $\nabla^2 V=0$, provided $\quad \mathrm{f}\lambda=C\int\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{-\frac{1}{2}}d\lambda.$

Examples.

(1) Interpret geometrically the equations

$$\mathrm{f}r=0, \mathrm{f}\theta=0, \mathrm{f}\phi=0, \mathrm{f}(\theta,\phi)=0, \mathrm{F}(r,\theta)=0, \mathrm{F}(r,\phi)=0.$$

(2) Prove that the quadric surfaces of revolution,

$$\frac{x^2+y^2}{\sec^2 u}+\frac{z^2}{\tan^2 u}=c^2 \ldots\ldots\ldots\ldots \text{(i.)},$$

$$\frac{x^2+y^2}{\operatorname{sech}^2 v}-\frac{z^2}{\tanh^2 v}=c^2 \ldots\ldots\ldots\ldots \text{(ii.)},$$

$$\frac{x^2}{\cos^2 w}-\frac{y^2}{\sin^2 w}=0 \ldots\ldots\ldots\ldots \text{(iii)},$$

are (i.) oblate spheroids, (ii.) hyperboloids of one sheet, (iii.) planes, forming a system of confocal orthogonal surfaces; and sketch the figure.

Prove that they intersect in the point

$$x=c\sec u\operatorname{sech} v\cos w,\quad y=c\sec u\operatorname{sech} v\sin w,$$
$$z=c\tan u\tanh v;$$

and that the generating lines of (ii.) are given by

$$\frac{x-c\sec u\operatorname{sech} v\cos w}{\operatorname{sech} v\sin(u\pm w)}=\frac{y-c\sec u\operatorname{sech} v\sin w}{-\operatorname{sech} v\cos(u\pm w)}=\frac{z-c\tan u\tanh v}{\pm\tanh v};$$

the angle between them being $2\cos^{-1}(\cos u\operatorname{sech} v)$.

Write down the corresponding equations for a system of confocal prolate spheroids and hyperboloids of two sheets.

(3) Prove that the equations

$$\frac{y^2}{\cosh^2\frac{1}{2}u}+\frac{z^2}{\sinh^2\frac{1}{2}u}=8a(a\cosh u-x) \ldots\ldots \text{(i.)},$$

$$\frac{y^2}{\cos^2\frac{1}{2}v}-\frac{z^2}{\sin^2\frac{1}{2}v}=8a(a\cosh v-x) \ldots\ldots \text{(ii.)},$$

$$\frac{y^2}{\sinh^2\frac{1}{2}w}+\frac{z^2}{\cosh^2\frac{1}{2}w}=8a(a\cosh w+x) \ldots\ldots \text{(iii.)}$$

represents a system of confocal paraboloids intersecting at right angles in the point

$$x = a(\cosh u + \cos v - \cosh w),$$

$$y = 4a \cosh \tfrac{1}{2}u \cos \tfrac{1}{2}v \sinh \tfrac{1}{2}w, \quad z = 4a \sinh \tfrac{1}{2}u \sin \tfrac{1}{2}v \cosh \tfrac{1}{2}w.$$

Determine A, B, C for this system, and the equations of the generating lines of (ii).

Prove also that $\nabla^2(u, v, \text{or } w) = 0$; as also in Ex. 2.

(4) Verify that $1/r$, ϕ, $\log \tan \frac{1}{2}\theta$, and their products are annihilated by the operator ∇^2.

Also $r^{-2}\cos\theta$, $r^{-2}\sin\theta\cos(\phi + a)$.

(5) Prove that, if A and B are the ends of a diameter of a sphere of radius a whose centre is at O, the function

$$V = \frac{1}{AP} - \frac{1}{BP} - \frac{1}{AB}\log\frac{AN + AP}{BN + BP},$$

where N is the foot of the perpendicular from P on AB, is annihilated by ∇^2; and also that $\partial V/\partial r = 0$, when $r = a$.

Give the physical interpretation of this result.

(6) Prove that

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2} + \dots + \frac{\partial^2 V}{\partial x_n^2}$$

$$= \frac{1}{W}\left\{\frac{\partial}{\partial r}\left(W\frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta_1}\left(\frac{W}{u_1}\frac{\partial V}{\partial \theta_1}\right) + \dots + \frac{\partial}{\partial \theta_{n-1}}\left(\frac{W}{u_{n-1}}\frac{\partial V}{\partial \theta_{n-1}}\right)\right\},$$

where

$$x_1 = r\sin\theta_1\sin\theta_2\dots\sin\theta_{n-1}, \quad u_1 = r^2,$$

$$x_2 = r\sin\theta_1\sin\theta_2\dots\cos\theta_{n-1}, \quad u_2 = r^2\sin^2\theta_1,$$

$$x_3 = r\sin\theta_1\sin\theta_2\dots\cos\theta_{n-2}, \quad u_3 = r^2\sin^2\theta_1\sin^2\theta_2,$$

....

$$x_{n-1} = r\sin\theta_1\cos\theta_2, \quad u_{n-1} = r^2\sin^2\theta_1\sin^2\theta_2\dots\sin^2\theta_{n-2},$$

$$x_n = r\cos\theta_1, \qquad W = r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\dots\sin\theta_{n-2}.$$

Examine the case of $n = 3$.

Prove also that

$$x_1\frac{\partial V}{\partial x_1} + x_2\frac{\partial V}{\partial x_2} + \dots + x_n\frac{\partial V}{\partial x_n} = r\frac{\partial V}{\partial r}.$$

148. *Quantics.*

A rational integral *homogeneous* algebraical function of the n^{th} degree in m variables $x, y, z, \ldots$ is defined to be "a function in which the sum of the indices of the variables in each term is constant and equal to n, the indices being positive integers"; such a function is called a *quantic*, and is denoted by $(x, y, z, \ldots)^n$.

Thus $$(x, y)^n, \text{ or } (a, b, c, \ldots)(x, y)^n$$
represents the *binary* quantic in x, y of the n^{th} degree
$$ax^n + nbx^{n-1}y + \tfrac{1}{2}n(n-1)cx^{n-2}y^2 + \ldots.$$

Denoting the general quantic by u, then
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \ldots = nu.$$

This is proved for each single term of the quantic, say
$$Ax^py^qz^r \ldots, \text{ where } p+q+r+\ldots = n;$$
for, denoting this term by v, then
$$x\frac{\partial v}{\partial x} = pv,\ y\frac{\partial v}{\partial y} = qv,\ z\frac{\partial v}{\partial z} = rv,\ \ldots;$$
and $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} + \ldots = nv.$$

More generally
$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \ldots\right)^2 u = n(n-1)u,$$
$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \ldots\right)^3 u = n(n-1)(n-2)u,$$
and so on; in which it is important to notice that the expression in brackets on the left hand side must be expanded by the Multinomial Theorem as if $x, \frac{\partial}{\partial x}, y, \frac{\partial}{\partial y}, \ldots$ were independent algebraical quantities, and then u supplied in the partial derivatives.

For if $\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\dots\right)^k u$ meant that the operation $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\dots$ was repeated k times, we should have

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\dots\right)^k u=n^k u.$$

The operator is therefore a particular case of the operator $x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+\dots$, denoted by Δ, and called by Klein the *polarizing operator*.

These theorems are proved by expanding the quantic

$$u_1=(x+hx,\ y+hy,\ z+hz,\ \dots)^n=(1+h)^n u$$

in the form (§ 126)

$$u_1=u+h\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\dots\right)u$$
$$+\frac{h^2}{2!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\dots\right)^2 u+\dots,$$

and equating coefficients of like powers of h.

These theorems are called *Euler's Theorems of Homogeneous Functions, or Quantics.*

A non-homogeneous function can be always made to appear homogeneous by the introduction of an appropriate factor to each term, consisting of the requisite power of some new variable, which may afterwards be interpreted as unity.

Thus the general equation of a conic section

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

can be made homogeneous, and a *ternary quadratic* in x, y, z, by writing it

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0;$$

and now x, y, z may be considered as trilinear coordinates; and replacing z by unity gives the ordinary Cartesian coordinates, with oblique axes.

*149. *Arbogast's Method of Derivation.*

Denoting $f(x+x't,\ y+y't,\ z+z't, \ldots)$ by f, then

$$\frac{df}{dt}=x'\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+z'\frac{\partial f}{\partial z}+\ldots;$$

so that the polarizations of $f(x, y, z, \ldots)$ by the successive operations of Δ are the coefficients of t, $t^2/2!$, $t^3/3!, \ldots$ in the expansion of f in powers of t.

Again, if we denote

$$a_0+a_1x+a_2\frac{x^2}{2!}+\ \ldots\ +a_r\frac{x^r}{r!}+\ \ldots \text{ by } y,$$

then

$$\frac{dFy}{dx}=\left(a_1+a_2x+\ \ldots\ +a_{r+1}\frac{x^r}{r!}+\ldots\right)F'y$$

$$=\left(a_1\frac{\partial}{\partial a_0}+a_2\frac{\partial}{\partial a_1}+\ldots+a_{n+1}\frac{\partial}{\partial a_n}+\ldots\right)Fy;$$

and thence generally

$$\frac{d^nFy}{dx^n}=\left(a_1\frac{\partial}{\partial a_0}+a_2\frac{\partial}{\partial a_1}+\ldots+a_{r+1}\frac{\partial}{\partial a_r}+\ldots\right)^n Fy.$$

Therefore, expanded in powers of x by Maclaurin's Theorem,

$$Fy=\sum\frac{x^n}{n!}\left(a_1\frac{\partial}{\partial a_0}+a_2\frac{\partial}{\partial a_1}+\ldots+a_{r+1}\frac{\partial}{\partial a_r}+\ldots\right)^n fa_0;$$

so that the coefficients of the powers of x are derived by the repeated operation of

$$a_1\frac{\partial}{\partial a_0}+a_2\frac{\partial}{\partial a_1}+\ldots+a_{r+1}\frac{\partial}{\partial a_r}+\ldots \text{ on } fa_0;$$

this process is called Arbogast's *Method of Derivation.*

In Arbogast's second method of Derivation

$$fy=f(a+bx+cx^2+dx^3+ex^4+\ldots),$$

is expanded in powers of x, in the form

$$fy=\Sigma\frac{x^n}{n!}\left(b\frac{\partial}{\partial a}+2c\frac{\partial}{\partial b}+3d\frac{\partial}{\partial c}+4e\frac{\partial}{\partial d}+\ldots\right)^n fa,$$

$$=A+Bx+Cx^2+Dx^3+Ex^4+\ \ldots,$$

and now we find

$$\begin{aligned}
A&=\mathrm{f}a\\
B&=\mathrm{f}'a\,.\,b\\
C&=\mathrm{f}'a\,.\,c+\mathrm{f}''a\,.\,\frac{b^2}{2!}\\
D&=\mathrm{f}'a\,.\,d+\mathrm{f}''a\,.\,bc+\mathrm{f}'''a\,.\,\frac{b^3}{3!}\\
E&=\mathrm{f}'a\,.\,e+\mathrm{f}''a(bd+\tfrac{1}{2}c^2)+\mathrm{f}'''a\,.\,\tfrac{1}{2}b^2c+\mathrm{f}''''a\,.\,\frac{b^4}{4!}
\end{aligned}$$

....

and so on, by a simpler mode of Derivation.

For example, applying this method to Ex. 3, p. 234,

$$\sqrt[x]{(1+x)}=\exp(1-\tfrac{1}{2}x+\tfrac{1}{3}x^2-\tfrac{1}{4}x^3+\tfrac{1}{5}x^4\dots);$$

here $a=1,\ b=-\tfrac{1}{2},\ c=\tfrac{1}{3},\ d=-\tfrac{1}{4},\ e=\tfrac{1}{5},\dots,$

and $\mathrm{f}a=\mathrm{f}'a=\mathrm{f}''a=\dots\dots\dots=e;$

thence $A=1,\ B=-\tfrac{1}{2},\ C=\tfrac{11}{24},\ D=-\tfrac{7}{16},\ E=\tfrac{2447}{5760},\dots.$

***150.** *Theorems of Lagrange, Laplace, and Burmann.*

Put $y=a+bx+cx^2+dx^3+\dots,$

and $\phi y=b+cx+dx^2+ex^3+\dots;$

so that $y=a+x\phi y.$

Then $\dfrac{\partial y}{\partial x}=\phi y+x\phi'y\dfrac{\partial y}{\partial x},\ \dfrac{\partial y}{\partial a}=1+x\phi'y\dfrac{\partial y}{\partial a};$

and thus $\dfrac{\partial y}{\partial x}=\phi y\dfrac{\partial y}{\partial a}.$

Then if $u=\mathrm{f}y,$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}\frac{\partial y}{\partial x}=\phi y\mathrm{f}'y\frac{\partial y}{\partial a}=\phi y\frac{\partial u}{\partial a};$$

and since

$$\frac{\partial}{\partial x}\left(\mathrm{F}y\frac{\partial y}{\partial a}\right)=\mathrm{F}'y\frac{\partial y}{\partial x}\frac{\partial y}{\partial a}+\mathrm{F}y\frac{\partial^2 y}{\partial x\partial a}=\frac{\partial}{\partial a}\left(\mathrm{F}y\frac{\partial y}{\partial x}\right),$$

therefore

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\phi y\mathrm{f}'y\frac{\partial y}{\partial a}\right)=\frac{\partial}{\partial a}\left(\phi y\mathrm{f}'y\frac{\partial y}{\partial x}\right)=\frac{\partial}{\partial a}\left\{(\phi y)^2\frac{\partial u}{\partial a}\right\};$$

and generally, by Induction,

$$\frac{\partial^n u}{\partial x^n}=\frac{\partial^{n-1}}{\partial a^{n-1}}\left\{(\phi y)^n\frac{\partial u}{\partial a}\right\}.$$

Now make $x=0$, therefore $y=a$, and $u=\mathrm{f}a$; and

$$u=\mathrm{f}y=\mathrm{f}a+\Sigma\frac{x^n}{n!}\,\frac{\partial^{n-1}}{\partial a^{n-1}}\{(\phi a)^n\mathrm{f}'a\};$$

this is called *Lagrange's Theorem*.

Thus, for instance, putting $\mathrm{f}y=\phi y$, we find

$$b=\phi a,\ c=\frac{1}{2!}\,\frac{db^2}{da},\ d=\frac{1}{3!}\,\frac{d^2b^3}{da^2},\ \ldots.$$

Again, suppose $y=\mu+\tfrac{1}{2}h(y^2-1)$;

then
$$y=\mu+\Sigma\frac{h^n}{n!}\,\frac{d^{n-1}}{d\mu^{n-1}}\left(\frac{\mu^2-1}{2}\right)^n.$$

But, on the condition that $y=\mu$ when $h=0$,

$$y=\frac{1}{h}\{1-\sqrt{(1-2\mu h+h^2)}\},\ \frac{\partial y}{\partial\mu}=(1-2\mu h+h^2)^{-\frac{1}{2}};$$

so that (ex. 6, p. 276)

$$P_n=\frac{1}{n!}\,\frac{d^n}{d\mu^n}\left(\frac{\mu^2-1}{2}\right)^n.$$

Laplace has given a slightly more general form to the theorem of Lagrange by putting

$$z=\mathrm{F}y=\mathrm{F}(a+x\phi y);$$

and now $\dfrac{\partial z}{\partial x}=\phi y\dfrac{\partial z}{\partial a}$, as before; and

$$\mathrm{f}z=\mathrm{f}(\mathrm{F}a)+\Sigma\frac{x^n}{n!}\,\frac{\partial^{n-1}}{\partial a^{n-1}}[\{\phi(\mathrm{F}a)\}^n\mathrm{f}'(\mathrm{F}a)].$$

Next put x or $(y-a)/\phi y=\psi y$; then

$$\mathrm{f}y=\mathrm{f}a+\Sigma\frac{(\psi y)^n}{n!}\,\frac{\partial^{n-1}}{\partial y^{n-1}}\left\{\frac{(y-a)^n\mathrm{f}'y}{(\psi y)^n}\right\}_{(y=a)};$$

called *Burmann's Theorem*, giving the expansion of $\mathrm{f}y$ in powers of any other function ψy, $y=a$ being a root of $\psi y=0$.

CHAPTER VI.

CURVES IN GENERAL.

151. *Curve Tracing in Cartesian Coordinates.*

A number of such curves have already been introduced previously, which presented no difficulty in tracing from their equations; a slight sketch will now be given of a systematic method of treatment, but for a more complete account the student is referred to the treatises on *Curve Tracing* by Frost or Woolsey Johnson.

Given the equation of a curve in the rational integral algebraical form of the implicit relation $f(x, y)=0$, first group the terms in *binary quantics* (§ 148), arranged in descending degree n, $n-1$, ..., 3, 2, 1, 0; and denote them by u_n, u_{n-1}, ... u_3, u_2, u_1, u_0; so that the equation of the curve becomes

$$u_n+u_{n-1}+ \dots +u_3+u_2+u_1+u_0=0\,;\dots\dots\dots\text{(A)}$$

then n is called the *degree* of the curve.

If u_0 does not vanish, the curve does not pass through the origin; but by changing the origin to a point on the curve we can make u_0 vanish, and now the new u_1, equated to zero, gives the tangent at this point.

Also $u_2+u_1=0$ is the equation of a conic section, osculating or having a contact of the second order (§ 118) at the point.

With rectangular axes, $C(x^2+y^2)+u_1=0$ will be the equation of a circle touching the curve; and if

$$u_1=Ax+By,\ u_2=ax^2+2hxy+by^2,$$

then this circle will be the circle of curvature, if

$$C=(aB^2-2hAB+bA^2)/(A^2+B^2).$$

Similarly $u_3+u_2+u_1=0$ will represent a *cubic* curve, having a contact of the *third* order (§ 118), and so on.

If u_1 also disappears, then $u_2=0$ represents two straight lines, real or imaginary, through the origin, which are tangents at this point; if they are real, the origin is a *double point*, and the curve crosses itself; if imaginary, the origin is a *conjugate point*, that is, an isolated point, the coordinates of which satisfy the equation of the curve.

If u_2 is a perfect square, the tangents are coincident and the origin is in general a *cusp* (§ 104).

If u_2 also vanishes, then $u_3=0$ denotes the tangents at the origin; so that if u_3 has three real linear factors, the origin is a *triple* point; and so on.

Generally to find the *multiple points* of a curve, that is the points where the curve crosses itself, consider the first derived equation of (A) (§ 125)

$$\frac{\partial \mathrm{f}}{\partial x}+\frac{\partial \mathrm{f}}{\partial y}\frac{dy}{dx}=0;$$

this gives in general a determinate value of dy/dx; except when $\partial\mathrm{f}/\partial x=0$, and $\partial\mathrm{f}/\partial y=0$, when the value of dy/dx becomes *indeterminate*, and the second, or third, ... derived equation must be employed to determine dy/dx; so that to determine the *multiple points* of a curve, we must find the values of x and y from the equations $\partial\mathrm{f}/\partial x=0$, $\partial\mathrm{f}/\partial y=0$, which also satisfy the equation of the curve $\mathrm{f}(x, y)=0$; and then a change of origin to one of these points will indicate its nature by inspection.

Newton's Analytical Parallelogram.

In tracing a curve in the neighbourhood of one of these singular points, taken as origin, it is important to know which terms in the equation can be neglected in comparison with a pair of others.

In Newton's Parallelogram a term of the equation Ax^my^n is represented graphically by a letter A at the point whose coordinates are m, n on a diagram; and now if another term Bx^py^q is placed on the diagram, the line AB will divide the diagram into two parts, such that a third term Cx^ry^s can be rejected if its representative point C lies on the side of AB remote from the origin, as being an infinitesimal of a higher order than the terms A and B, which have been retained.

When all the other terms of the equation can be rejected in comparison with Ax^my^n and Bx^py^q, then

$$Ax^my^n+Bx^py^q=0, \text{ or } Ax^{m-p}+By^{q-n}=0$$

will give a close approximation to the curve.

Examples.

(1) Determine the tangents at the origin of

(i.) $x^2(x+y)-a^2(x-y)=0$;

(ii.) $x^2y^2-a^3(x+y)=0$;

(iii.) $x^2y^2-a^2(x^2-y^2)=0$;

(iv.) $x^4+y^4+6ax^2y-8ay^3=0$;

(v.) $x^4+3ax^2y+2axy^2-ay^3=0$;

(vi.) $x^2+y^2=a^2,\ 2ax,\ 2ay$;

(vii.) $x^3+y^3=a^3,\ a^2x,\ a^2y,\ ax^2,\ 3axy,\ ay^2$;

(viii.) $x^4+y^4=a^4,\ a^3x, \ldots,\ a^2x^2,\ a^2xy,\ ax^3,\ ax^2y$;

(ix.) $x^5+y^5=a^5, a^4x, a^3x^2, a^3xy, a^2x^3, ax^4, ax^3y, ax^2y^2, \ldots$;

and sketch the curves near O, by means of Newton's Parallelogram.

(2) Determine the equation of the tangent of a circle, parabola, ellipse, hyperbola, and generally of a conic at any point, by changing the origin to the point and back again.

(3) Prove that chords of a conic, which subtend a right angle at a fixed point O in the conic, pass through a fixed point on the normal at O.

152. *Asymptotes.*

To determine the nature of the curve at an infinite distance from the origin, consider the geometrical interpretation of the equation $u_n=0$, which represents n straight lines, real or imaginary, through the origin.

The real straight lines will approximate to the nature of the curve at infinity, and will be parallel to the rectilinear *asymptotes*, if asymptotes exist.

A rectilinear *asymptote* is defined to be "a straight line, at a finite distance from the origin, to which a branch of the curve continually approaches, and ultimately at an infinite distance becomes indefinitely near."

The equation of an asymptote will therefore be of the form $y=mx+c$, when $y=mx$ is the equation of one of the straight lines represented by $u_n=0$; the asymptotes of a hyperbola are familiar instances.

The problem of finding an asymptote is then, from the implicit relation $\mathrm{f}(x, y)=0$, to expand y by *reversion of series* (§ 85) in descending powers of x, in the form

$$y=mx+c+px^{-1}+qx^{-2}+\dots$$

Substituting this expansion of y in terms of x in the equation $\mathrm{f}(x, y)=0$, and treating the resulting equation as an identity, then by equating to zero the coefficients of x^n, x^{n-1}, x^{n-2}, ..., sufficient equations are obtained to determine m, c, p, q, ...

Then m determines the direction and c the position of the asymptote, while p, or in its absence q, ... determines the side of the asymptote on which the curve lies; for this reason it is generally useful to expand y in descending powers of x as far as three terms.

We notice that if u_{n-1} is absent, and $y-mx$ is not a repeated factor of u_n, then $c=0$, and the corresponding asymptote passes through the origin.

Thus, the general equation of a conic section, of excentricity e, and semi-latus-rectum l, is

$$y^2=2lx+(e^2-1)x^2,$$

when the origin is at a vertex; and putting

$$y=mx+c+px^{-1}+...,$$

we find $m^2=e^2-1$, $c=l/m$, $p=-n^2/2m$, ...

The asymptotes are therefore real only for hyperbolas, in which $e > 1$; and their equation will then be

$$y=\pm\sqrt{(e^2-1)}\left(x+\frac{l}{e^2-1}\right).$$

When $e=1$, the conic is a parabola, and the asymptote is given by $m=0$, $c=\infty$; so that it lies at an infinite distance, and does not therefore satisfy the definition of an asymptote.

If u_n has a factor x, then to determine the corresponding asymptote, we must expand x in descending powers of y, in the form

$$x=c'+p'y^{-1}+q'y^{-2}+...;$$

or we may put, in general,

$$x=m'y+c'+p'y^{-1}+...,$$

and determine m', c', p', ... as before.

Sometimes the expansion of y in descending powers of x or x in powers of y must be written

$$y = Ax^2 + mx + c + px^{-1} + \dots,$$
$$x = A'y^2 + m'y + c' + p'y^{-1} + \dots,$$

and then $y = Ax^2 + mx + c$, or $x = A'y^2 + m'y + c'$ is called a *parabolic asymptote*; and so on.

When it is possible to obtain y *explicitly* in terms of x, or x in terms of y (§ 13) from the *implicit* relation $f(x, y) = 0$, the asymptotes are then readily determined by expanding by the Binomial Theorem and other algebraical operations in descending powers.

Thus if $$x^3 + y^3 = a^3,$$
then $y = \sqrt[3]{}(a^3 - x^3) = -x(1 - a^3x^{-3})^{\frac{1}{3}} = -x + \frac{1}{3}a^3x^{-2} + \dots;$
or $x = \sqrt[3]{}(a^3 - y^3) = -y(1 - a^3y^{-3})^{\frac{1}{3}} = -y + \frac{1}{3}a^3y^{-2} + \dots.$

Also if $x = a$ makes $y = \infty$, or $y = b$ makes $x = \infty$, then $x - a = 0$ is an asymptote; and so also is $y - b = 0$.

For instance, if the equation of the curve is
$$(a/x)^2 + (b/y)^2 = 1,$$
then $y^2 = b^2x^2/(x^2 - a^2)$, and $x^2 = a^2y^2/(y^2 - b^2)$;
so that $x \pm a = 0$, and $y \pm b = 0$ are asymptotes.

The preceding considerations are in general sufficient for tracing a curve whose Cartesian equation is given, but considerations of symmetry are also useful; thus if *even* powers only of x appear in the equation, the curve is symmetrical right and left of the axis of y: if *even* powers only of y appear, the curve is symmetrical above and below the axis of x, as if reflected in Ox.

If x and y are involved symmetrically, then $x - y = 0$ is an axis of symmetry, in which the curve is reflected, and on being doubled along this line the two halves of the curve will come into coincidence.

Examples.

(1) Determine the asymptotes of the following curves, and draw them:—

(i.) $x^2-y^2=a^2$; (ii.) $(x/a)^2-(y/b)^2=1$;
(iii.) $y^2=2ax+x^2$; (iv.) $x(x^2+y^2)-ay^2=0$;
(v.) $x^2y+xy^2=a^3$; (vi.) $x^3-xy^2+ay^2=0$;
(vii.) $x^2y^2=a^2(x^2+y^2)$, or $a^2(x^2-y^2)$;
(viii.) $x^4-y^4-a^2xy=0$; (ix.) $xy(x^2-y^2)=a^4$;
(x.) $xy(x^2+y^2)=a(x^3+y^3)$;
(xi.) $\alpha\beta\gamma=c^3$, where α, β, γ are the perpendiculars on the sides of an equilateral triangle.

(2) Determine the asymptotes of the curves in the preceding set of examples (p. 317), and draw them.

(3) Determine the equation of the cubic curve which has the asymptotes

$$2x-y+3a=0,\ x+y-3a=0,\ x+y+a=0,$$

and cuts the axis of x at O at an angle $\tan^{-1}(-2)$.

(4) Draw, for different values of c, the curves
(i.) $x^2+2cxy+y^2=1$;
(ii.) $x^2+2cxy+x^2=1-c^2$,
equivalent to
$c=xy+\surd(1-x^2)\surd(1-y^2)$, or $xy+\surd(x^2-1)\surd(y^2-1)$;
(iii.) $x^4+2cx^2y^2+y^4=1$;
(iv.) $x^4+2cx^2y^2+y^4=1-c^2$.

(5) Draw the curve $(x^2-a^2)^2+(y^2-b^2)^2=c^4$,
(i.) $c^4>a^4+b^4$, (ii.) $a^4+b^4>c^4>a^4$, (iii.) $c^4=a^4$,
(iv.) $a^4>c^4>b^4$, (v.) $c^4=b^4$, (vi.) $b^4>c^4$.

(6) Draw the curves
(i.) $\tan^{-1}x/a+\tan^{-1}y/b=\tan^{-1}c$;
(ii.) $\tanh^{-1}x/a+\tanh^{-1}y/b=\tanh^{-1}c$;
(iii.) $(\tan x)^2+(\tan y)^2=\frac{1}{3}$, or 3.

153. *Polar Coordinates.*

In discussing properties of a curve connected with straight lines radiating from any origin O, it is convenient to change to polar coordinates (r, θ) by putting (§ 22)

$$x = r\cos\theta,\ y = r\sin\theta.$$

Substituting in the rational integral equation (A), p. 315,

$$u_0 r^{-n} + (A\cos\theta + B\sin\theta)r^{-n+1}$$
$$+ (a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)r^{-n+2} + \ldots = 0;$$

an equation of the n^{th} degree in r^{-1}; so that a straight line cuts a curve of the n^{th} degree in n points, some pairs of which however may be imaginary.

Denoting by $r_1, r_2, r_3, \ldots r_n$ the roots in r of this equation and their *harmonic mean* by r, then

$$r^{-1} = (r_1^{-1} + r_2^{-1} + \ldots + r_n^{-1})/n = -(A\cos\theta + B\sin\theta)/nu_0,$$

or
$$Ar\cos\theta + Br\sin\theta + nu_0 = 0,$$
$$Ax + By + nu_0 = 0,\ \text{or}\ u_1 + nu_0 = 0,$$

the equation of a straight line, the locus of a point P, such that OP is the harmonic mean of $OP_1, OP_2, \ldots OP_n$, where $P_1, P_2, \ldots P_n$ are the n points in which the straight line cuts the curve; this straight line is called the *polar line* of O, by analogy with the polar line of a conic section, with which it coincides when $n = 2$.

The straight line $u_1 + u_0 = 0$ is the locus of P when

$$r^{-1} = r_1^{-1} + r_2^{-1} + \ldots + r_n^{-1}.$$

In the same way the *polar conic* of O,

$$u_2 + (n-1)u_1 + \frac{n(n-1)}{1.2}u_0 = 0,$$

is the locus of P when

$$\Sigma\left(\frac{1}{OP} - \frac{1}{OP_r}\right)\left(\frac{1}{OP} - \frac{1}{OP_s}\right) = 0;$$

and so on.

Writing, as usual, u for $1/r$, then since for the points of intersection with the curve,

$$\Sigma u = -(A\cos\theta + B\sin\theta)/nu_0,$$

therefore $$\Sigma(d^2u/d\theta^2 + u) = 0.$$

But $$d^2u/d\theta^2 + u = c\operatorname{cosec}^3\phi,$$

by § 93, where c denotes the curvature, and ϕ the radial angle at which the vector OP cuts the curve; so that

$$\Sigma c\operatorname{cosec}^3\phi = 0,$$

at the points of intersection of OP with the curve. (Dr. Routh, *Quarterly J. of Math.*, xxiv., p. 257.)

We have found that in a central field of force (§ 84), the acceleration to the centre

$$P = h^2u^2(d^2u/d\theta^2 + u),$$

so that the orbit is a straight line, and $d^2u/d\theta^2 + u = 0$, when $P = 0$; but the orbit is *concave* to the origin when P and therefore $d^2u/d\theta^2 + u$ is positive; and *convex* when they are negative; and at a *point of inflexion*, where the curve changes from *concavity* to *convexity*, or *vice versa*, $d^2u/d\theta^2 + u$ vanishes and changes sign.

Definition.—A curve is said to be *concave* with respect to a point or line when it lies on the same side of its tangent as the point or line in the neighbourhood of the point of contact; and *convex* when it lies on the opposite side of its tangent; and at a *point of inflexion* the curve crosses its tangent.

In interpreting the above, some of the ρ's must be negative, and we shall take ρ as positive or negative according as the curve is concave or convex to the origin.

Incidentally we deduce that the three points of inflexion on a cubic lie in a straight line.

154. *Equation of the Chord, Tangent, Asymptote, and Normal of a Curve in Polar Coordinates.*

It is convenient to employ u, the reciprocal of r (§ 23), and now the equation of a straight line can be written

$$u = A\cos\theta + B\sin\theta\,;$$

or, more generally,

$$u = A\cos(\theta-\alpha) + B\sin(\theta-\alpha),$$

equivalent in Cartesian coordinates to

$$1 = A(x\cos\alpha + y\sin\alpha) + B(y\cos\alpha - x\sin\alpha),$$

an equation of the first degree is x and y, and therefore the equation of a straight line.

To find the equation of the chord of the curve, whose equation is $u=\mathrm{f}\theta$, which passes through the two points whose vectorial angles are $\alpha\pm\beta$, we must determine A and B from the equations

$$\mathrm{f}(\alpha+\beta) = A\cos\beta + B\sin\beta,\ \mathrm{f}(\alpha-\beta) = A\cos\beta - B\sin\beta\,;$$

and therefore

$$A = \frac{\mathrm{f}(\alpha+\beta)+\mathrm{f}(\alpha-\beta)}{2\cos\beta},\ B = \frac{\mathrm{f}(\alpha+\beta)-\mathrm{f}(\alpha-\beta)}{2\sin\beta}\,;$$

so that the equation of the chord is

$$u = \frac{\mathrm{f}(\alpha+\beta)+\mathrm{f}(\alpha-\beta)}{2\cos\beta}\cos(\theta-\alpha) + \frac{\mathrm{f}(\alpha+\beta)-\mathrm{f}(\alpha-\beta)}{2\sin\beta}\sin(\theta-\alpha)\ldots(\text{i.})$$

To determine the tangent where $\theta=\alpha$, put $\beta=0$; then

$$\mathrm{lt}\frac{\mathrm{f}(\alpha+\beta)+\mathrm{f}(\alpha-\beta)}{2\cos\beta} = \mathrm{f}\alpha,\ \mathrm{lt}\frac{\mathrm{f}(\alpha+\beta)-\mathrm{f}(\alpha-\beta)}{2\sin\beta} = \mathrm{f}'\alpha\,;$$

and therefore the equation of the tangent at $\theta=\alpha$ is

$$u = \mathrm{f}\alpha\,.\cos(\theta-\alpha) + \mathrm{f}'\alpha\,.\sin(\theta-\alpha)\ldots\ldots\ldots(\text{ii.})\,;$$

so that, if $OP=1/\mathrm{f}\alpha$, then $Ot=-1/\mathrm{f}'\alpha$ (fig. 10);
and looking along OP from O, then Ot must be drawn to the right if $\mathrm{f}'\alpha$ is negative; to the left if positive.

Suppose $\mathrm{f}\alpha=0$, but $\mathrm{f}'\alpha$ is finite; the point of contact is then at an infinite distance from O, but the tangent remains at a finite distance, and the tangent is therefore an asymptote, its equation being

$$u = f'\alpha \,.\, \sin(\theta-\alpha) \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii.)}.$$

If $f'\alpha=\infty$, the asymptote passes through O, and its equation is given by $\theta=\alpha$.

The equation of the normal will be

$$u = f\alpha \,.\, \cos(\theta-\alpha) - \frac{(f\alpha)^2}{f'\alpha}\sin(\theta-\alpha) \ldots\ldots\ldots\text{(iv.)};$$

for it is the equation of a straight line through the point of contact of the tangent, where $\theta=\alpha$; and this straight line is perpendicular to the tangent, as is readily seen by changing to Cartesians.

When the equation of the curve is given in the form $r=F\theta$, the equation of the normal will be found to be

$$\frac{1}{r} = \frac{\cos(\theta-\alpha)}{F\alpha} + \frac{\sin(\theta-\alpha)}{F'\alpha} \ldots\ldots\ldots\ldots\ldots\text{(v.)};$$

so that $Og = F'\alpha$, if $OP = F\alpha$ (fig. 10).

For instance, the equation of the chord, tangent, and normal, of the conic $lu = 1 + e\cos\theta$, are

$$lu = e\cos\theta + \sec\beta\cos(\theta-\alpha),\quad lu = e\cos\theta + \cos(\theta-\alpha).$$

$$lu = \frac{1+e\cos\alpha}{\sin\alpha}\{e\sin\theta + \sin(\theta-\alpha)\}.$$

Examples.

(1) Find the asymptotes of

(i.) $r = a\sec\theta$, $b\operatorname{cosec}\theta$, $a\sec\theta + b\operatorname{cosec}\theta$, $a\sec(\theta-\alpha)$, $b\operatorname{cosec}(\theta-\beta)$, $a\sec(\theta-\alpha) + b\operatorname{cosec}(\theta-\beta)$.

(ii.) $r = a\tan\theta$, $a\cot\theta$.

(iii.) $r = a(\sec\theta - \cos\theta)$, $a(\sec\theta + \tan\theta)$; and determine their equations in x and y.

(iv.) $r = \sec 2\theta$, $\operatorname{cosec} 2\theta$, $\sec 3\theta, \ldots \sec n\theta$, $\operatorname{cosec} n\theta$.

(v.) $r^2 = a^2\sec 2\theta$, $a^2\operatorname{cosec} 2\theta$.

(vi.) $r = a + b\operatorname{cosec}\theta$ (the *conchoid* of Nicomedes).

(vii.) $r = a/\theta$, $a\theta/(\theta-\alpha)$, $a\theta^2/(\theta^2-\alpha^2)$, $a\sin m\theta/\sin\theta$.

(viii.) $l/r = \pm 1 + \sec\beta\cos\theta$ (the hyperbola).

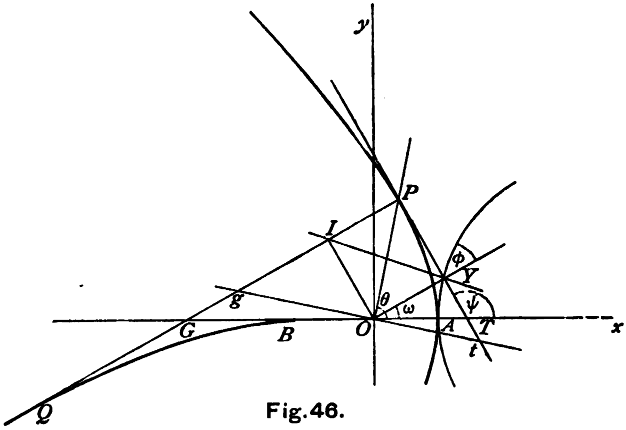


Fig. 46.

155. *Pedal Curves.*

The locus of Y, the foot of the perpendicular on the tangent of a curve drawn from the origin O, is called the *pedal* of the curve with respect to O, and O is called the *pole* of the pedal.

Thus with respect to a focus, the pedal of an ellipse or hyperbola is the auxiliary circle, and the pedal of a parabola is the tangent at the vertex of the parabola.

If OYP is a rigid right angle, of which OY passes through O, and YP touches the curve, then I, the foot of the perpendicular from O on the normal at P, is the *centre of instantaneous rotation* of the right angle (§ 21); so that IY is the normal of the locus of Y.

Since IY is a diameter of the circle described on OP as diameter, it follows that the envelope of circles described on the variable vector OP as diameter is the pedal with respect to O, the locus of Y.

This can easily be generalized for the case of a rigid angle PYP' touching two fixed curves, at P and P'; the centre of instantaneous rotation will be at I, the point of intersection of the normals at P and P', and therefore IY is the normal of the locus of Y; and since the circle circumscribing the triangle PYP' has the same normal at Y, it follows that the locus of Y is the envelope of these circles, described round the triangle PYP'.

Returning to the pedal of the curve (fig. 46), then since the angle OYI is equal to the angle OPI in the same segment, it follows that the pedal curve cuts OY at the same angle as the curve cuts OP, or the pedal and the curve have the same radial angle ϕ (§ 22) at corresponding points.

Therefore, denoting OY by p, and the angle AOY by ω,

$$\cot\phi=\frac{YP}{OY}=\frac{dp}{pd\omega},\text{ so that } YP=\frac{dp}{d\omega}.$$

We denote OI by q, so that the polar coordinates of I are (q, ψ) and $q=dp/d\omega$, $\psi=\omega+\frac{1}{2}\pi$; and since Q, the centre of curvature at P, is the point of contact of the normal PI of the curve AP, or of the tangent PI of the evolute BQ at Q, therefore the locus of I is the pedal of the evolute with respect to O, and $IQ=dq/d\psi=d^2p/d\omega^2$.

Therefore the radius of curvature at P,

$$ds/d\omega=\rho=p+dq/d\psi=p+d^2p/d\omega^2=p+qdq/dp.$$

The equation of the tangent YP being

$$x\cos\omega+y\sin\omega=p;$$

and of the normal PI being

$$x\cos\psi+y\sin\psi=q \text{ or } -x\sin\omega+y\cos\omega=dp/d\omega;$$

it is therefore derived from the tangent by differentiating with respect to ω.

The equation of the normal to the evolute at Q will therefore be $-x\cos\omega - y\sin\omega = d^2p/d\omega^2$;
so that the coordinates of Q are

$$-\sin\omega\frac{dp}{d\omega} - \cos\omega\frac{d^2p}{d\omega^2},\ \cos\omega\frac{dp}{d\omega} - \sin\omega\frac{d^2p}{d\omega^2}.$$

The relation $p = \mathrm{f}\omega$, connecting p and ω for the curve AP, is called its *tangential polar* equation; and $p = \mathrm{f}\omega$ is the polar equation of the pedal curve AY, with p and ω as polar coordinates; and then $q = \mathrm{f}'(\omega + \frac{1}{2}\pi)$ will be the polar equation of the pedal of the evolute BQ.

The original curve AP is the *envelope* (§ 105) of

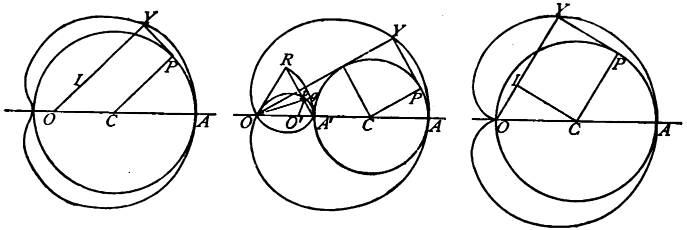
$$x\cos\omega + y\sin\omega = \mathrm{f}\omega;$$

and with reference to the curve AY, the curve AP is called the *first negative pedal* of AY with respect to O. Thus the first negative pedal of a circle is an ellipse or hyperbola, according as the pole is inside or outside the circle; the first negative pedal of a straight line is a parabola.

For instance the directions of motion of the parts of a rope in contact with a moving pulley are at any instant tangents to a conic, of which a focus is at the instantaneous centre of rotation of the pulley.

We may take the pedal of the pedal, positive or negative, with respect to the pole O, any number of times n, and then we obtain a curve called the n^{th} pedal, positive or negative.

Suppose rays of light proceeding from O are incident on a reflecting curve AP; the reflected ray will pass through Z on OY, produced so that $OZ = 2OY$, and will be parallel to YI, and therefore normal to the locus of Z; the envelope of the reflected rays, called the *katakaustic* (§ 105), is therefore the evolute of the locus of Z, a curve double the size of the pedal curve AY.



(i) (ii) (iii)

Fig. 47

156. *Limaçons, the Pedals of a Circle.*
Consider the pedal of a circle, centre C and radius a, with respect to any point O, where $OC=b$.

Then $$p=OY=IY+OI=a+b\cos\omega,$$ the equation connecting the polar coordinates p and ω of the locus of Y, and this curve is called a *limaçon.*

The limaçon is called a *conchoid* of a circle, because it is described by producing the vector of the circle $r=b\cos\theta$ a constant distance a.

The equation of the limaçon may be written
$$p=\pm a+b\cos\omega,$$
corresponding to parallel tangents of the circle; so that the chord of the limaçon through O is of constant length $2a$.

If $b<a$, O is inside the circle, and the pedal consists of a single oval curve (fig. 47, i.). This oval has points of inflexion, if $b>\frac{1}{2}a$.

If $b>a$, O is outside the circle, and the pedal is looped, having a double point O (fig. 47, ii.).

If $b=a$, O is on the circumference, and $p=a(1+\cos\omega)$, the equation of a *cardioid* (fig. 47, iii.).

If in fig. 8 we fix the bar PQR, and move the cross OQR, then any point fixed in OQ or OR will describe a limaçon.

157. *Orthoptic and Isoptic Curves.*

If two tangents PR, QR to the curve APQ intersect at a constant angle a, in the point R, the locus of R is called an *isoptic* curve of the curve APQ; and if the angle a is a right angle, the locus of R is called the *orthoptic* curve.

Thus the isoptic curve of a circle is a concentric circle the orthoptic locus of an ellipse or hyperbola is a circle sometimes called the *director* circle; the orthoptic locus of a parabola is its directrix, while the isoptic locus of a parabola is a confocal hyperbola.

The equation of the *isoptic* locus for tangents inclined at an angle a is obtained by eliminating ω between

$$x\cos(\omega+\tfrac{1}{2}a)+y\sin(\omega+\tfrac{1}{2}a)=\mathrm{f}(\omega+\tfrac{1}{2}a),$$
$$x\cos(\omega-\tfrac{1}{2}a)+y\sin(\omega-\tfrac{1}{2}a)=\mathrm{f}(\omega-\tfrac{1}{2}a);$$

and $a=\tfrac{1}{2}\pi$ gives the *orthoptic* locus.

The normal RI of the isoptic locus at R will pass through I the point of intersection of the normals at the points P, Q; since I is the centre of instantaneous rotation of the constant angle PRQ.

Examples.

(1) Prove that the equation of the pedal of an ellipse with respect to the centre is

$$p^2=a^2\cos^2\omega+b^2\sin^2\omega, \text{ or } (x^2+y^2)^2=a^2x^2+b^2y^2.$$

Show that the pedal has points of inflexion when $b^2/a^2<\tfrac{1}{2}$, or $e^2>\tfrac{1}{2}$; and that, when $b=0$, the pedal reduces to two circles.

(2) Prove that the pedal of a parabola with respect to the vertex is the *cissoid*

$$p=a(\sec\omega-\cos\omega), \text{ or } y^2=x^3/(a-x).$$

(3) Prove that the isoptic locus of a parabola is a hyperbola, of excentricity $\sec a$; and explain how the two branches of the hyperbola are formed.

(4) The isoptic locus of an ellipse is given by

$$(x^2+y^2-a^2-b^2)^2=4\cot^2\alpha(a^2y^2b^2+x^2-a^2b^2).$$

(5) The isoptic locus of a cycloid is a trochoid.

(6) The orthoptic locus of $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ is $\frac{1}{2}\sqrt{2}\,a\cos 2\theta$.

(7) The orthoptic locus of the cardioid is composed of a circle and a limaçon (Wolstenholme).

(8) Prove that if $\rho_1, \rho_2, \rho_3, \ldots$ are the radii of curvature of the envelopes of the sides of a polygon, whose sides are $a, b, c, \ldots$ then $a\rho_1+b\rho_2+c\rho_3+\ldots$ is equal to twice the area of the polygon.

(9) Prove that the polar equation of the curve OP of § 119, in the neighbourhood of O, is

$$r=2\rho\theta+\tfrac{4}{3}\rho\rho'\theta^2+\ldots.$$

(10) Prove that the pedal of an involute of a circle, with respect to the centre of the circle, is a spiral of Archimedes (fig. 36).

Apply this to the theory of a weighing machine, showing the weight on a dial provided with equal graduations, when the body is weighed against a fixed weight, suspended by a rope which wraps on the involute of a circle.

158. *Roulettes.*

When a curve, carrying a point P fixed to it, rolls on a straight line (or any given curve), the path traced out by the point P is called the roulette of P with respect to the straight line (or given curve).

Thus, when a circle rolls on a straight line, the roulette of a point on the circumference is a *cycloid*, and the roulette of any other point fixed in the plane of the circle is a *trochoid* (§ 21).

An *involute* of a curve (§ 95) is thus the roulette of a point on a straight line which rolls on the curve.

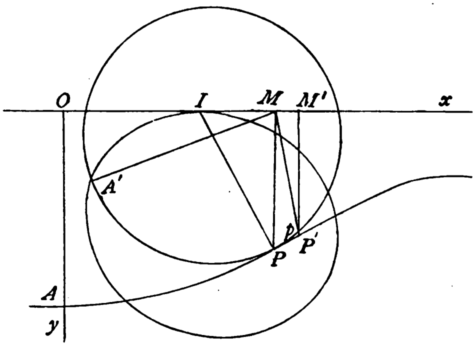


Fig. 48

A remarkable analogy, pointed out by Steiner, exists between the roulette of a point with respect to a straight line and the pedal of the rolling curve with respect to the point as pole. Steiner's Theorems assert that

(i.) The length of the arc of the roulette is equal to the length of the corresponding arc of the pedal;

(ii.) The area bounded by an arc of the roulette, the ordinates at the ends of the arc, and the straight line on which the curve rolls is twice the area bounded by the corresponding arc of the pedal and the vectors from the origin to the ends of the arc.

For, if AP is the roulette of the point P when the curve is rolled on the straight line Ox (fig. 48), and if PM is the perpendicular from P on Ox, the tangent at I to the rolling curve, then relatively to P the locus of M is the pedal of the rolling curve with respect to P; and therefore relatively to M the locus of P is the same curve, and the subnormal IM of the roulette is the q or $dp/d\omega$ of the rolling curve, or the polar subnormal of its pedal.

We may suppose the pedal $A'P$ rolled on the roulette AP, so that M is always vertically over P if Ox is horizontal; and the pedal, if loaded so that the centre of gravity is at M, will rest in neutral equilibrium on the roulette, provided the friction is sufficient, or else that teeth are cut, to prevent slipping.

The arc AP of the roulette will then be equal to the corresponding arc $A'P$ of the pedal, which is Steiner's first theorem.

The arc s in the roulette is therefore the same function of y as in the pedal of p.

Also if the pedal is rolled into a consecutive position so that M comes to M', and the point p of the pedal comes into contact with the point P' of the roulette, then the element $MM'P'P$, which is the increment of area of the roulette, is ultimately double the element MPp, which is the increment of area of the pedal, or

$$\text{lt}(\text{area } MM'P'P)/\text{area } MPp = 2;$$

and therefore, by integration, the area $OMPA$ of the roulette is double the area $A'MP$ of the pedal, which is Steiner's second theorem.

Examples.

(1) Draw the figures and compare the arcs and areas of the following pairs of curves, the first curve of a pair being fixed, and the second rolling on it, so that its pole describes a straight line Ox (or Oy); thus illustrating Steiner's Theorems.

(i.) The parabola $y^2 = 2lx$, and the spiral $r = l\theta$.

(ii.) The circle $x^2 + y^2 = a^2$, and the circle $r = a \cos \theta$.

(This principle is employed in the parallel motion of Deleuil's air pump. Deschanel, *Physics.*)

(iii.) The ellipse $(x/a)^2+(y/b)^2=1$,
and $r=b\cos(b\theta/a)$ or $a\cos(a\theta/b)$.

(iv.) The hyperbola $(x/a)^2-(y/b)^2=1$,
and $r=a\cosh(a\theta/b)$ or $b\sinh(b\theta/a)$.

(v.) The exponential curve $y=be^{x/c}$,
and the reciprocal spiral $r=c/\theta$.

(vi.) The cycloid and the cardioid.

(vii.) The trochoid and the limaçon.

(viii.) The straight line $y=x\tan\alpha$,
and the equiangular spiral $r=ae^{\theta\tan\alpha}$.

(ix.) The catenary $y=a\cosh x/a$,
and the straight line $r=a\sec\theta$.

(x.) The modified catenary $y=b\cosh x/a$,
and the Cotes's spiral $r\cos(b\theta/a)=b$.

(xi.) The sinusoid $y=b\cos x/a$,
and the Cotes's spiral $r\cosh(b\theta/a)=b$.

(xii.) The sinusoid $y=a-ae\cos(x/b)$,
and the ellipse $l/r=1+e\cos\theta$.

(xiii.) The hyperbola $xy=c^2$, and the spiral $r^2\theta=\frac{1}{2}c^2$.

(xiv.) The semicubical parabola $ay^2=x^3$,
and the spiral $r\theta^3=8a$.

(xv.) The curves $(x/a)^m=(y/b)^n$ and $r^{m-n}\theta^m=c^{m-n}$.

(xvi.) The curves $x^my^n=a^{m+n}$, and $r^{m+n}\theta^m=c^{m+n}$.

Rectify these curves, when possible; and prove that s is the same function of y in the first curve as of r in the second; also that ydy/dx and ydx/dy, the subnormal and subtangent, are the same functions of y, as $dr/d\theta$ or $r^2d\theta/dr$, the polar subnormal and subtangent are functions of r.

(2) Prove that the roulette of the pole of
$$l/r=1+\sec\alpha\cos(\theta\sin\alpha),\text{ or }1+\operatorname{sech}\alpha\cosh(\theta\sinh\alpha),$$
with respect to a straight line, is a circle.

(3) Prove that the curvature of the roulette with respect to a straight line is $d \sin \phi / dp$, where ϕ is the radial angle and p the perpendicular on the tangent from the pole on the rolling curve.

(4) Prove that if the centre of an ellipse is fixed at a distance b from a plane, and the ellipse is rolled on the plane, the point of contact will describe the Cotes's spiral $r \cosh (ae\theta/b) = ae$.

159. *Centrodes.*

When a moving plane figure slides or turns on another plane, which may be considered fixed, then a point I in the moving plane can always be found which has no velocity; this point I is called the *centre of instantaneous rotation* (§ 21), and the relative motion of the two planes is assigned by the position of I and by the angular velocity n of the moving plane round I.

The point I will in the general case describe a curve in the moving plane and a curve in the fixed plane, and the motion of the moving plane will be given by rolling the first curve on the second; so that any point carried by the rolling plane will describe a roulette of the first curve with respect to the second curve; these curves described by I are now called the *centrodes* of the relative motion of the two planes.

Another point J can always be found of which the *acceleration* is zero; and now the acceleration of any other carried point P at a distance r from J will be composed of component accelerations n^2r towards J and $\dot{n}r$ perpendicular to JP, $\dot{n}$ denoting the angular acceleration dn/dt.

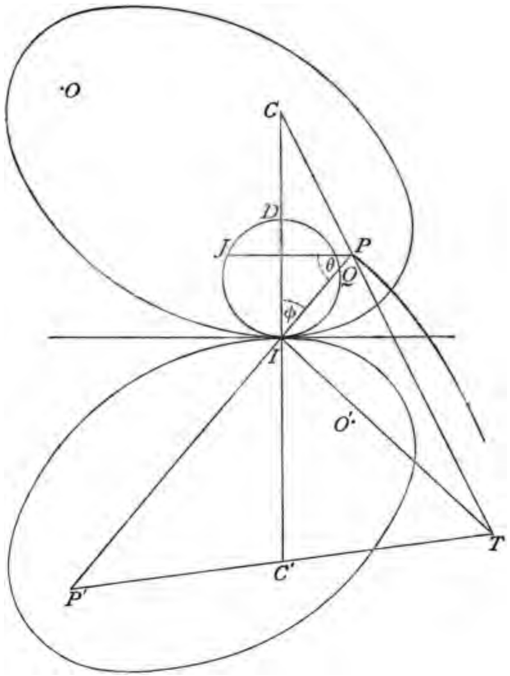


Fig. 49

Therefore the resultant acceleration of any point, such as P, makes the same angle $\alpha = \cot^{-1}(\dot{n}/n^2)$ with JP; and the lines of resultant acceleration at any instant are equiangular spirals of radial angle α, the magnitude of the acceleration at distance r from J being $n^2 r \operatorname{cosec} \alpha$.

Denoting by R the radius of curvature PP' of the trajectory of P, by v the velocity of P, and by θ the angle IPJ, then (§ 92) the normal acceleration of P,

$$v^2/R = n^2 r \cos\theta - \dot{n} r \sin\theta = n^2 r \operatorname{cosec} \alpha \sin(\alpha - \theta).$$

The points whose trajectories have zero curvature, and which are therefore at this instant describing straight paths (as M in fig. 48), or rather are passing through points of inflexion on their trajectories, are obtained by putting $\theta=\alpha$, and therefore lie on a circle IQJ passing through I and J, called the *circle of inflexions* (the circle PMI in fig. 48); and this circle will touch the centrodes at I, since the acceleration at M is in the normal.

Now if IP meets this circle in Q, then the curvature of the trajectory of P is

$$1/R=n^2r\sin(\alpha-\theta)/v^2\sin\alpha$$
$$=n^2.PQ/v^2=PQ/PI^2.$$

Thus $$PP'=PI^2/PQ,$$

$$IP'=PI^2/PQ-PI=PI.IQ/PQ,$$

or $$\frac{1}{IP'}=\frac{PQ}{PI.IQ}=\frac{1}{IQ}-\frac{1}{PI},$$

or $$\frac{1}{PI}+\frac{1}{IP'}=\frac{1}{IQ}=\frac{\sec\phi}{ID},$$

where ID is the diameter of the circle of inflexions, and where ϕ now denotes the angle between IP and the common normal of the centrodes at I.

A point P inside the circle of inflexions will thus describe a trajectory *convex* with respect to I, but *concave* if the point is outside this circle; so that the rolling centrode, if loaded so that its centre of gravity is at P, will be in stable or unstable equilibrium with IP vertical according as P is inside or outside this circle.

The diameter ID of the circle of inflexions is inferred by placing the carried point P for a moment at C, the centre of curvature of the rolling centrode at I, when it is easily seen that the centre of curvature of the roulette through C will be at C', the corresponding centre of curvature at I of the fixed centrode; and now, with $\phi=0$,

$$\frac{1}{ID}=\frac{1}{CI}+\frac{1}{IC'}=\frac{1}{\rho}+\frac{1}{\rho'},$$

or $ID=\rho\rho'/(\rho+\rho')$, where ρ, ρ' denote the radii of curvature CI, IC' of the rolling and fixed centrodes, reckoned positive when the centrodes are convex to each other.

Now $$\frac{1}{PI}+\frac{1}{IP'}=\sec\phi\left(\frac{1}{\rho}+\frac{1}{\rho'}\right),$$

and the symmetry of this relation shows that P is the centre of curvature of the roulette of P' with respect to the former moving centrode; also that CP, $C'P'$ intersect in a point T on IT the perpendicular to IP.

When a curve rolls symmetrically on an equal curve, the roulette of any point will be similar to the pedal of the curve with respect to P, but enlarged to twice the scale; the reason being that the reflexion of the carried point P in the tangent at I is a fixed point.

Suppose for instance, as drawn in fig. 49, that the fixed and rolling centrodes are equal ellipses, and that the carried point P is at one of the foci; the roulette of P will be a circle, and the centre of curvature P' will lie at a focus of the fixed ellipse.

We can pivot these ellipses at the other foci O and O', and now revolve them in contact with each other; teeth may be cut to prevent slipping, and the revolving foci P and P' may be connected by a link to prevent separation. This mechanism is sometimes employed, and the ellipses roll on each other as if connected with a crossed parallelogram of bars OP, PP', $P'O'$, $O'O$.

160. *The Area of a Roulette.*

As I moves along the rolling and fixed centrodes with the same velocity ds/dt, the moving plane will turn in the time dt through an angle $d\omega$, which is the sum of the curvatures ds/ρ and ds/ρ' of the equal arcs ds of the rolling and fixed centrodes; so that

$$n=\frac{d\omega}{dt}=\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)\frac{ds}{dt}.$$

The normal PI of the roulette of P now sweeps out an area, denoted by (PI), at a rate which, by Guldin's Theorem generalized (§ 62), will be measured by the product of PI and of the component velocity of the middle point of PI perpendicular to PI.

This component velocity, being the arithmetic mean of the velocities of I and P in the same direction, is equal to

$$\tfrac{1}{2}\cos\phi(ds/dt)+\tfrac{1}{2}n\,.\,PI;$$

so that

$$\frac{d(PI)}{dt}=\tfrac{1}{2}PI\cos\phi\frac{ds}{dt}+\tfrac{1}{2}n\,.\,PI^2=\tfrac{1}{2}p\frac{ds}{dt}+\tfrac{1}{2}\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)PI^2\frac{ds}{dt}.$$

Of these two terms, the first is the rate at which PI sweeps out polar area on the rolling centrode with respect to the pole P (§ 56); and the second represents the growth of the M.I. round P of the perimeter of the rolling centrode, supposing the perimeter replaced by a wire of variable density $\tfrac{1}{2}(c+c')$, the arithmetic mean of the curvatures $c=1/\rho$ and $c'=1/\rho'$ of the rolling and fixed centrodes.

Thus if the rolling centrode returns to its original position after one or more complete revolutions, the area of the roulette of P exceeds the sum of the areas of the rolling and fixed centrodes by the moment of inertia of this perimeter of variable density of the rolling centrode.

By taking the carried point P at G, the centre of gravity of this perimeter, we obtain the minimum moment of inertia (§ 64), and therefore the roulette of minimum area; and the area (P) of the roulette of P will exceed the area (G) of the roulette of G by $M.GP^2$, where M is the mass distributed on the perimeter.

But $d\omega/dt$ denoting the angular velocity of the rolling centrode,
$$M=\int\tfrac{1}{2}(c+c')ds=\int\tfrac{1}{2}d\omega=n\pi,$$
for n complete revolutions of the rolling centrode; so that
$$(P)-(G)=n\pi.GP^2.$$

161. *Theorems of Holditch, Elliott, Leudesdorf, and Kempe.*

A simple relation connects the areas (P), (A), (R) swept out by three points P, A, R in a straight line on a bar, which we may take to be the bar of the planimeter of fig. 45, carried round by the rolling centrode, and making n complete revolutions.

For if GK is the perpendicular from G on AP produced,

then
$$(P)-(A)=n\pi(GP^2-GA^2)=n\pi(KP^2-KA^2)$$
$$=n\pi b(KP+KA),$$

and
$$(A)-(R)=n\pi c(KA+KR),$$

since
$$AP=b,\ RA=c;\ \text{so that}$$
$$\frac{(A)-(R)}{c}-\frac{(P)-(A)}{b}=n\pi(KR-KP)=n\pi.PR,$$

or
$$(b+c)(A)-b(R)-c(P)=n\pi bc(b+c);$$

this is called *Holditch's Theorem.*

If the closed contours described by A and P lie entirely outside each other, the bar AP can only oscillate between two extreme positions, and $n=0$.

In using the planimeter, the pivot O is generally fixed outside the contour described by P, and the joint A oscillates on the arc of a circle of radius a, so that

$$(A)=0 \text{ and } n=0; \text{ and thus } b(R)+c(P)=0,$$

attending to the sign of the area.

Thus if P describes a circle of radius r not enclosing O, the motion of the bars OA, AP is similar to that of the beam and connecting rod of a steam engine; and now $(R)=-\pi r^2 c/b$; this is independent of the length of the beam, and is therefore the same for the ordinary direct action steam engine, as applied in the locomotive engine, where A oscillates in a straight line.

A new form of Planimeter has been lately brought out, in which the end A of the bar AP is constrained to move in a straight or curved slot, while the pencil P is carried round the area to be measured; but now the area is registered by the motion of a wheel W which can slide and turn on a graduated round bar CD, projecting at right angles to AP from a point C, which we may take to be the middle point of AP.

The rate at which AP sweeps out area is the product of AP into the component velocity of C perpendicular to AP; and this component velocity is equal to the velocity with which the wheel W slides along CD; the wheel being supposed to roll, but not to slide on the paper; in this manner the sliding motion of the roller R on the paper of Amsler's Planimeter is obviated.

Mr. Elliott has generalized Holditch's theorem by showing that the bar PAR may be replaced by an elastic thread, which stretches, but keeps the ratio b/c constant, when a similar Theorem holds.

Mr. Leudesdorf has extended the theorem to the relation connecting the areas of the roulettes (A), (B), (C) of any three points A, B, C with the area (P) of the roulette of any fourth point P; and he finds that

$(P)-x(A)-y(B)-z(C)=n\pi$ times the rectangle on the segments of a chord through P of the circle circumscribing the triangle ABC, when x, y, z denote the ratios of the triangles PBC, PCA, PAB to the triangle ABC.

For $\quad (P)-(A)=n\pi(GP^2-GA^2), \ldots;$

so that $(P)-x(A)-y(B)-z(C)$

$$=n\pi(GP^2-x\,.\,GA^2-y\,.\,GB^2-z\,.\,GC^2)$$
$$=n\pi(x\,.\,BC^2+y\,.\,CA^2+z\,.\,AB^2)$$

which can be shown to lead to Mr. Leudesdorf's result; and reduces to Holditch's Theorem when P lies in a side BC.

Mr. Kempe points out that the locus of P, for which (P) is zero, is a circle; and the locus of P, for which (P) is constant, is a concentric circle, exceeding in area the first circle by an amount proportional to (P); but that if $n=0$, this system of concentric circles must be replaced by a system of parallel straight lines.

(*Messenger of Mathematics*, vol. vii., 1878.)

162. *The Envelope of a Carried Line.*

Similar theorems hold for the envelope of a carried line Nx, fixed in the moving centrode and carried round by it.

The point of contact N on the envelope is the foot of the perpendicular IN drawn from I on the carried line; and now if we denote by y the ordinate IN of the point I of the rolling centrode with respect to the line Nx, by x the corresponding abscissa on the line Nx, and by S the length of the arc described by N; then

$$\frac{dS}{dt}=\frac{dx}{dt}+y\frac{d\omega}{dt},$$

so that $$S-x=\int y\,d\omega=\int \tfrac{1}{2}(c+c')y\,ds;$$
or S the length of the arc of the envelope exceeds x the projection of the corresponding arc of the rolling centrode on the carried line by the moment about the carried line of the same distribution of density $\frac{1}{2}(c+c')$ as before.

Carried lines which have envelopes of the same length will therefore touch a circle, with the centre of gravity G of the perimeter as centre.

Again, if ϕ' denotes the angle between IN and the common normal of the centrodes at I, the radius of curvature R' of the envelope of the carried line at N is given by

$$\begin{aligned} R'&=\frac{dS}{d\omega}=\frac{dx}{d\omega}+y=\cos\phi'\frac{ds}{d\omega}+y\\ &=Q'I+IN=Q'N, \end{aligned}$$

supposing NI meets in Q' the circle $IQ'D'$, which is the reflexion of the circle of inflexions in the common tangent of the centrodes at I.

Therefore, for all carried lines passing through D', $R'=0$, and the envelope has a cusp lying on this circle, called for this reason the *circle of cusps ;* while in addition the circle is the locus of the centres of curvature of all carried lines, the centre of curvature Q' being obtained by dropping the perpendicular $D'Q'$ on IN.

The normal NI of the envelope will now sweep out area (NI) at a rate

$$\frac{d(NI)}{dt}=\tfrac{1}{2}y\left(\frac{dx}{dt}+\frac{dS}{dt}\right)=y\frac{dx}{dt}+\tfrac{1}{2}y^2\frac{d\omega}{dt};$$

so that the area swept out by the normal NI exceeds the area swept out on the rolling centrode by the ordinate NI by the M.I. of the perimeter, of variable density $\frac{1}{2}(c+c')$, round the carried line.

Thus from Theorem II., p. 128, it follows that if two carried lines intersect at right angles in P, the sum of the areas of their envelopes exceeds the area of the roulette of P by the area of the rolling centrode.

Carried lines which sweep out equal areas are such that the M.I. of the perimeter round them is the same; they are therefore *equimomental* lines, and therefore tangents of an *equimomental ellipse;* and by varying the area we obtain a system of equimomental confocal ellipses, by well known theorems.

(Besant, *Roulettes and Glissettes;*
Kempe, *Messenger of Mathematics; Nature,* 1878;
Darboux, *Bulletin des Sciences Mathématiques,* 1878;
Larmor, *Nature,* 1881; *Proc. Cam. Phil. Soc.,* 1890;
Minchin, *Uniplanar Kinematics,* 1882.)

163. When the fixed centrode becomes a straight line, $c'=0$; and the area swept out by the normal PI (fig. 48) in a complete revolution of the rolling curve is the same as that swept out by the ordinates MP, and is therefore double the area of the pedal of the rolling curve with respect to P.

Now, if the rolling curve is coated with matter of linear density $\frac{1}{2}c$ per unit of length, the mass will be

$$\int \tfrac{1}{2}c\,ds = \int_0^{2\pi} \tfrac{1}{2}d\omega = \pi;$$

and therefore the area of the roulette of P will exceed the area of the roulette of G, the centre of gravity of this wire, by $\pi . GP^2$; and consequently the area of the pedal of the rolling curve with respect to P will exceed the area of the pedal with respect to G by $\frac{1}{2}\pi . GP^2$.

Thus, for instance, the pedal of a circle with respect to its centre being the circle itself, will have an area πa^2; and consequently the area of the limaçon $p=a+b\cos\omega$ will be $\pi a^2+\frac{1}{2}\pi b^2$, while the area of the corresponding trochoid will be $2\pi a^2+\pi b^2$; and with $a=b$, the area of the cardioid $p=a(1+\cos\omega)$ will be $\frac{3}{2}\pi a^2$, and of the cycloid $3\pi a^2$.

In a roulette with respect to a straight line,

$$\frac{1}{R-N}+\frac{1}{N}=\frac{\sec\phi}{\rho},$$

or

$$NR-N^2=\rho R\cos\phi,$$

or

$$\frac{1}{R}=\frac{1}{N}-\frac{\rho\cos\phi}{N^2}.$$

If the rolling curve is an ellipse and the carried point at a focus, then (fig. 48)

$$\rho\cos\phi=N(2a-N)/a=2N-N^2/a,$$

so that

$$\frac{1}{R}+\frac{1}{N}=\frac{1}{a}.$$

If this roulette sweeps out a surface by revolution round the fixed straight line, the curvature of this surface will be everywhere $1/a$, a constant; a soap-bubble film of revolution will assume the shape of this surface, or a surface similarly generated by the roulette of the focus of a hyperbola or parabola.

For instance, the roulette of the focus of a parabola is a catenary and we obtain the catenoid of fig. 16.

Again, the roulette of the pole of the involute of a circle with respect to a straight line is a parabola; and the roulette of the centre of the rectangular hyperbola $r^2\cos 2\theta=a^2$ is a curve in which $R=\frac{1}{2}N$, or $Ry=\frac{1}{2}a^2$, an *elastica* or *lintearia*.

164. *Epicycloids and Hypocycloids.*

These curves are the roulettes of a point P on the circumference of a circle which rolls on the outside or inside of a fixed circle.

Let O denote the centre and a the radius of the fixed circle, C the centre and c the radius of the rolling circle, and I the point of contact of the circles; then IP is the normal of the roulette of P, because I is the centre of instantaneous rotation of the rolling circle (fig. 50).

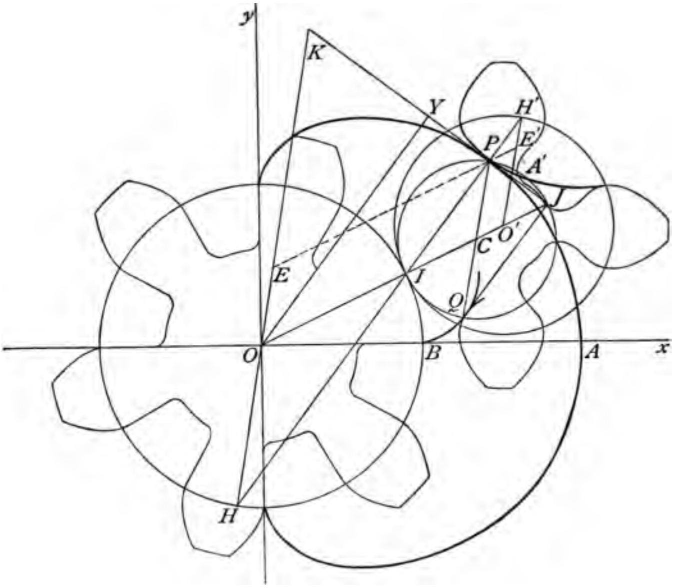


Fig. 50.

Draw the diameter PCQ of the rolling circle, and suppose Q originally in contact with the fixed circle at B, and that P is then at A; A is then an apse and B a cusp of an epicycloid (§ 104).

Denoting the angle xOI by θ (radians), then the arc $IB = a\theta = \text{arc } IQ$, so that the angle $ICQ = a\theta/c$; and the coordinates of P in terms of θ, for the epicycloid, are

$$x = (a+c)\cos\theta + c\cos(1+a/c)\theta,$$
$$y = (a+c)\sin\theta + c\sin(1+a/c)\theta;$$

and for a hypocycloid, change c into $-c$.

Thence we find, for the epicycloid,

$$\frac{ds}{d\theta} = 2(a+c)\cos\frac{a\theta}{2c},$$

and integrating, the arc AP

$$= s = 4\frac{c}{a}(a+c)\ \sin\frac{a\theta}{2c};$$

while the $\quad$ arc $BQ = 4\frac{c}{a}(a+c)\operatorname{vers}\frac{a\theta}{2c}$.

With the notation of p and ω of § 154,

$$p = (a+2c)\cos(a\theta/2c), \text{ and } \omega = (1+a/2c)\theta;$$

so that $\quad p = (a+2c)\cos\{a\omega/(a+2c)\}$,

the polar equation of the pedal of an epicycloid, of the form $r = b\cos m\theta$.

Also $\quad PY = -\frac{dp}{d\omega} = a\sin\frac{a\omega}{a+2c} = a\sin\frac{a\theta}{2c};$

so that $\quad r^2 = OY^2 + PY^2$

$$= (a+2c)^2\cos^2\frac{a\theta}{2c} + a^2\sin^2\frac{a\theta}{2c}$$

or $\quad r^2 - a^2 = 4c(a+c)\cos^2\frac{a\theta}{2c} = \frac{4c(a+c)}{(a+2c)^2}p^2,$

of the form $\quad r^2 - a^2 = (1-m^2)p^2,\ m = a/(a+2c),$

the relation connecting p and r in the epicycloid; proving incidentally that the roulette of the centre of an epicycloid with respect to a straight line is an ellipse.

If $m = 0$, then $c = \infty$, and the curve becomes the involute of a circle, in which $p^2 = r^2 - a^2$.

Again $$s = 4\frac{c}{a}(a+c)\sin\frac{a\omega}{a+2c},$$

of the form $s = l\sin m\psi$, with ψ for ω, l for $4c(a+c)/a$, and m for $a/(a+2c)$, as in § 104.

And $$\rho = \frac{ds}{d\omega} = \frac{4c(a+c)}{a+2c}\cos\frac{a\omega}{a+2c} = \frac{rdr}{dp} = (1-m^2)p.$$

165. *The Teeth of Wheels.*

By cutting teeth on wheels we can make them engage and transmit power without slipping; we thus secure the condition called *perfect roughness* by the theoretical mathematician; perfect smoothness between two bodies on the other hand is sought practically by the interposition of wheels.

Suppose O' to be the fixed centre of a wheel of radius $O'I = a'$, which is to be made to revolve in contact with the wheel of centre O and radius a, without slipping.

If the circle, centre C and radius c, rolls inside this circle, centre O' and radius a', and describes the hypocycloid $A'P$, then if the epi- and hypo-cycloids AP and $A'P$ start with the vertices A and A' in contact, the two curves will roll and slide on each other (fig. 50) so that the common normal at P passes through I, and therefore the constant velocity ratio of the wheels is maintained; for this reason the wheels of teeth are shaped by epi- and hypo-cycloids.

Only a small portion of each curve in the neighbourhood of a cusp is made use of to form a tooth; and the tooth is completed on the circle O' by a portion of an epicycloid, and on the circle O by a portion of a hypocycloid, each described by the rolling of a circle of the same radius c'.

For instance, if $c' = \frac{1}{2}a$, then the hypocycloid in the circle O is given by $x = 0$, $y = a \sin \theta$; so that the hypocycloid degenerates into a straight line, a diameter; and the inside portion of the tooth is straight and radial.

Any number of change wheels of a lathe may be made to work together indiscriminately, provided the teeth are shaped by rolling circles of the same radius c.

In fig. 50 we have taken $a/a' = \frac{6}{4}$, and $c = c' = \frac{1}{2}a$; and four teeth on the circle O engage with six teeth on O'.

When we make c and c' infinite, the teeth are shaped by *involutes* of the circle O and O'; involute teeth have the advantage of preserving the velocity ratio of the wheels unchanged for variable distances between the centres of the wheels, and are employed in rolling mills.

When the radius a' is made infinite, the corresponding wheel becomes converted into a *rack*, and the teeth on the rack are shaped by cycloids.

Sometimes to ensure greater regularity of working, *helical* teeth are employed, and now the tooth of one helix on the cylinder of radius a works with the tooth of the helix of equal pitch on the cylinder of radius a'; so that when one helix rolls on another of equal pitch on a parallel axis, any point of the helix describes an epicycloid. (MacCord, *Kinematics*; G. B. Grant, *Odontics*).

166. *The Double Generation of Epi- and Hypo-cycloids.*

Produce PI both ways to meet the circles O and O' again in H and H', and draw EPE' parallel to OO' to meet OH and $O'H'$ in E and E'.

Then E and E' are the centres of circles, of radii $a+c$ and $a'-c$, which touch each other at P and the circles O and O' at H and H'; so that the epicycloid AP and the hypocycloid $A'P$ can be described by the rolling of these circles on the circles O and O'; we thus perceive that there is a *double mode of generation of the epicycloid and hypocycloid.*

Relatively to the circle O', any point on the circumference of the circle O describes an epicycloid; and a hypocycloid if it is made to roll inside the circle O'.

The envelope of the diameter QCP is another epicycloid, described by a rolling circle of radius $\frac{1}{2}c$, as is readily perceived when we notice that the point of contact of the envelope is at the foot of the perpendicular drawn from I on PQ.

The envelope of any other carried straight line, say parallel to PQ, will be a parallel curve to the epicycloid, the envelope of PQ, and will therefore have an epicycloidal evolute.

167. *Epi- and Hypo-Trochoids, or Bicircloids.*

A point fixed in CP at a distance kc from C will describe a curve, given by

$$x=(a+c)\cos\theta+kc\cos(1+a/c)\theta,$$

$$y=(a+c)\sin\theta+kc\sin(1+a/c)\theta;$$

these curves are called epi- or hypo-trochoids, and sometimes *bicircloids.*

The relative orbit of two planets is a bicircloid, if the planets describe circles round the Sun; figures of the relative orbits of the Earth and the different planets, drawn mechanically by Mr. Perigal, are given in Proctor's *Geometry of Cycloids.*

For if α, β denote the (mean) distances from the Sun, and n, m the mean motions, the relative orbit is given by

$$x=\alpha\cos nt+\beta\cos mt,\ y=\alpha\sin nt+\beta\sin mt;$$

where $n^2\alpha^3=m^2\beta^3$, by Kepler's Third Law (§ 179); so that

$$a+c=\alpha,\ kc=\beta,\ \text{and } 1+a/c=m/n=(\alpha/\beta)^{\frac{3}{2}};$$

or $$a=\alpha-\beta^{\frac{3}{2}}/\alpha^{\frac{1}{2}},\ c=\beta^{\frac{3}{2}}/\alpha^{\frac{1}{2}}.$$

When $c=a$, or $m=2n$, the bicircloids are *limaçons* (§ 156), as is otherwise geometrically obvious. Thus if we assume that the period of Mars is two years, the relative orbit of the Earth and Mars will be a limaçon.

Examples.

(1) Prove that if an epicycloid rolls on a straight line, the centre will describe an ellipse.

(2) Prove that, if a helix rolls on a straight line, any point on the helix will describe a cycloid.

(3) The shadow of a helix on a plane perpendicular to its axis, thrown by parallel rays of light, is a trochoid.

Find when the trochoid will be a cycloid.

(4) The path of a steamer turning uniformly in a current will be a trochoid relatively to the land.

(5) Show that a variable velocity ratio of two wheels can be attained by cutting teeth in equiangular spirals of the same radial angle.

(6) If an equiangular spiral rolls on a circle, the pole will describe an involute of a circle.

(7) Show that the path of the Moon relatively to the Sun has no points of inflexion.

(8) The isoptic locus of an epicycloid is an epitrochoid (Wolstenholme, *Proc. London Math. Soc.*, vol. iv.).

If the isoptic locus passes through the centre, its equation is of the form $r = b \sin\{a\theta/(a+2c)\}$.

(9) Prove that the epicycloid has (i.) one cusp when $c = a$, $2a$, $3a$, $4a$, ...; and then $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$; (ii.) two cusps when $2c = a$, $3a$, $5a$, ...; and then $m = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$; and draw them.

(10) Prove that the equation of the *Tricusp* hypocycloid can be written

(i.) $p = \frac{1}{3} a \cos 3\omega$, (ii.) $s = \frac{8}{9} a \sin 3\psi$,

(iii.) $(x^2+y^2)^2 + \frac{8}{3} ax(x^2-3y^2) + 2a^2(x^2+y^2) - \frac{1}{3}a^4 = 0$,

(iv.) $r^4 + \frac{8}{3} ar^3 \cos 3\theta + 2a^2r^2 - \frac{1}{3}a^4 = 0$.

Prove that if the tangent at P meets the Tricusp again in Q and R,

(i.) the length $QR = \frac{4}{3} a$,

(ii.) the tangents at Q and R intersect at right angles in a point T on the inscribed circle of the Tricusp,

(iii.) the normals at P, Q, R intersect in a point N on the circumscribed circle,

(iv.) NT passes through the centre.

168. *Inversion and Inverse Curves.*

When the vector OP of a curve is produced to Q, so that OQ is inversely proportional to OP, or $OQ = c^2/OP$, then the locus of Q is called an *inverse curve* of the locus of P with respect to the origin O, or with respect to the circle of the centre O and radius c; and c is then called the *constant of inversion.*

Thus if $u = \mathrm{f}\theta$ is the polar equation of the locus of P, then $r = c^2 \mathrm{f}\theta$ is the polar equation of the inverse curve, the locus of Q, with respect to the origin O.

The inverse of a circle (or sphere) is another circle (or sphere); except when the circle (or sphere) passes through the centre of inversion, when the inverse is a straight line (or plane).

For if OPQ meets a circle (or sphere) in P and Q, then $OP.OQ$ is constant and equal to OT^2 for an external origin O, where OT is a tangent to the circle (or sphere); so that a circle (or sphere) is its own inverse with respect to any origin O; and varying the constant of inversion gives a similar curve (or surface); in this case another circle (or sphere).

Maxwell calls the rectangle on the segments of a chord through O made by the circle (or sphere) the *power* of the circle (or sphere) with respect to O; and now if the distances from O to the centre of the circle or sphere and of its inverse are denoted by a and a', and the radii by b and b', then a^2-b^2, $a'^2-b'^2$ are the *powers* of the circles or spheres, and $\dfrac{a'}{a}=\dfrac{b'}{b}=\dfrac{c^2}{a^2-b^2}=\dfrac{a'^2-b'^2}{c^2}$.

Any curve Pp and its inverse Qq with respect to O will cut the vector OPQ at supplementary radial angles; for since $OP.OQ=c^2=Op.Oq$, a circle can be drawn through $PQqp$, and therefore the angles pPQ, pqQ are supplementary, and Pp, Qq are ultimately the tangents at P and Q, when p and therefore q are brought up close to P and Q.

Or, otherwise, denoting the radial angles by ϕ and ϕ',

$$\cot\phi=d\log r/d\theta=-d\log u/d\theta=-\mathrm{f}'\theta/\mathrm{f}\theta=-\cot\phi',$$

$$\cot\phi+\cot\phi'=0, \text{ or } \phi+\phi'=\pi;$$

so that the radial angles are supplementary.

Any two curves and their inverses will therefore cut at the same angle.

A small element of area is consequently unaltered in shape by inversion, but the alteration of linear scale is

$$r'/r = c^2/r^2 = r'^2/c^2.$$

So also in space, an element of volume will be undistorted by inversion, but will be changed in volume in the scale c^6/r^6 or r'^6/c^6.

If accented letters refer generally to the inverse curve, then

$$Ot' = c^2/Og,\ Og' = c^2/Ot;$$

$$p' = c^2/Pg,\quad p = c^2/Qg';$$

$$Pt\,.\,Qt' = c^2\sec^2\phi;$$

$$\frac{r'}{\rho'} + \frac{r}{\rho} = \frac{dp}{dr} + \frac{dp'}{dr'} = \frac{dr\sin\phi}{dr} + \frac{dr'\sin\phi}{dr'} = 2\sin\phi;$$

and the circle of curvature obviously inverts into the circle of curvature of the new curve.

Lines of curvature (§ 130) will correspond on a surface and its inverse; and if a circle of principal curvature is drawn, the normal plane of this circle and the sphere through the circle and origin of inversion O will invert into the sphere and plane through the corresponding circle of principal curvature.

When the Cartesian equation of a plane curve is given in rectangular coordinates x and y, the equation of the inverse curve with respect to the origin O is obtained by writing $c^2x/(x^2+y^2)$ for x, and $c^2y/(x^2+y^2)$ for y.

Thus the inverse of the parabola $y^2=px$ with respect to the vertex is, writing a for c^2/p, the *cissoid*

$$x(x^2+y^2)=ay^2,\text{ or } y^2=x^3/(a-x);$$

or in polar coordinates,

$$r=a(\sec\theta-\cos\theta),$$

the locus of t in a circle, for an origin on the circumference (ex. 11, p. 42).

169. *Mechanical Invertors and exact Parallel Motion.* When P describes a given curve, the point Q can be made to describe an inverse curve by means of the mechanical invertors, invented by Peaucellier and Hart.

Peaucellier's motion consists of a rhombus LP, PM, MQ, QL, formed of four links of equal length, jointed at L, M, P, Q, and of two other equal links OL, OM, pivoted at a fixed point O (fig. 51).

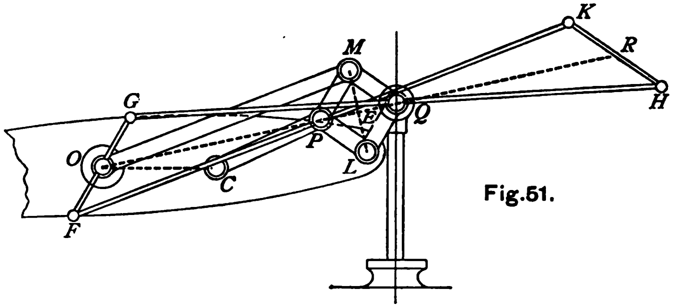


Fig.51.

Then, in whatever way the link motion is displaced by the motion of P,

$$OP\,.\,OQ = OE^2 - EP^2 = OL^2 - LP^2 = c^2,$$

a constant, so that P and Q describe inverse curves.

For instance, if P is compelled to describe an arc of a circle by a link CP, pivoted at a fixed point C, and of length $CP = OC$, then Q will move in a straight line perpendicular to OC.

A *parallel motion* is the name given in Mechanism to an arrangement of bars intended to guide a point Q, the head of a piston rod, in a straight line; and this is effected with exactness by Peaucellier's arrangement, but only approximately by Watt's parallel motion.

Hart's parallel motion accomplishes the same purpose with four bars, while Peaucellier's requires six.

A jointed parallelogram of rods, *FGHK*, is taken, and the longer rods are crossed; any fixed point *O* in one of the rods *FG* is taken, and *OPQR* is drawn parallel to *FH* or *GK*, to meet *FK* in *P*, *GH* in *Q*, and *HK* in *R* (fig. 51).

Then *P*, *Q*, *R* are also fixed points in the bars, such that when again opened into a parallelogram, *OPRQ* will form another parallelogram, the sides of which are parallel to the diagonals of *FGHK*, and the area of which bears a constant ratio to the area of *FGHK*; so that $OP \,.\, OQ$ is constant; and thus the parallelogram *FGHK*, when crossed, will act as an invertor, *P* and *Q* describing inverse curves when *O* is fixed.

We can show Hart's parallel motion working in conjunction with Peaucellier's by drawing *FG* parallel to *PM* or *LQ*, *GQH* parallel to *OL*, and *FPK* parallel to *OM*; and now *O*, *P*, *R*, *Q* are the middle points of the bars of the crossed parallelogram *FGHK*.

We may join *LR* and *MR* by bars, and now the two rhombuses *LPMQ*, *OLRM* are said to make a complete Peaucellier cell.

When *P* and *Q* are inside the cell, the cell is called *positive* (fig. 51), where it is shown acting as the parallel motion of a beam engine.

But when *P* and *Q* are outside, it is called a *negative* cell (fig. 52); and now it forms a compact mechanism for drawing not only straight lines, but circles of very large radius, as required in Architecture.

(Kempe, *How to Draw a Straight Line.*)

While *Q* describes a straight line and *P* a circular arc, the point *R* will describe a curve whose polar equation is of the form

$$r = a \sec\theta \pm b\cos\theta,$$

the inverse of a conic section with respect to a vertex.

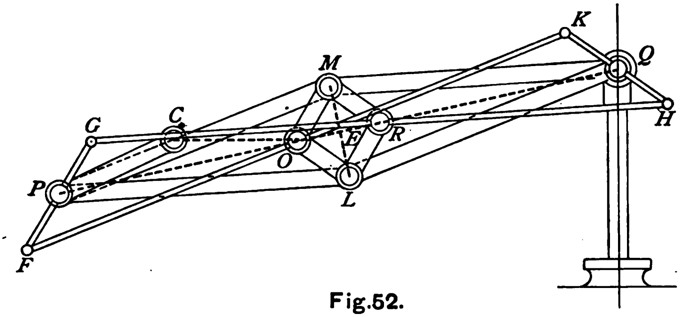


Fig. 52.

If equal bars *QB*, *BE* are pivoted at *Q*, *B*, and a fixed point *E* in the straight line described by *Q*, then, as in fig. 8, any point *D* in the bar *QB* will describe an arc of an ellipse; and the normal *DI* will pass through *I*, fixed in *EB* produced, so that $EI = 2EB$.

Any complicated system of linkwork in mechanism can always be analysed into *three bar motion*, such as *OA*, *AP*, *PC* in fig. 45, when a bar *PC* is pivoted about a fixed point at *C*; thus representing the beam, connecting rod, and crank in a beam engine; the beam being practically of infinite length in the direct action engine.

If *OA*, *CP* produced intersect in *I*, then *I* is the instantaneous centre of rotation of *AP*; and for uniform velocity of the crank *P*, the velocity of *A* will be a maximum when *OC* and *AP* intersect in a point *J*, the foot of the perpendicular from *I* on *AP*.

A point *Q* in *AP* will describe a figure of 8 curve; Watt's parallel motion consists essentially of a three bar motion, such as *AO*, *AP*, *PC*, by means of which the head of the piston rod *Q* is guided for a short distance on each side of the point of inflexion on this figure of 8.

The curves described by any point Q on the connecting rod AP in three bar motion have been studied by Roberts, Cayley, and Clifford; any such curve being capable of a triple generation by means of bars (Clifford, *Kinematic*).

When a crossed parallelogram is employed, as in figs. 51, 52, and in fig. 49, the relative motion of the short bars FG and HK is imitated by rolling an ellipse with foci F, G, on an equal ellipse with foci H, K; and the relative motion of the long bars GH, FK is imitated by rolling a hyperbola on an equal hyperbola; and relatively to one bar any point carried by the opposite bar will describe the pedal of an ellipse or hyperbola.

170. *Polar Reciprocals.*

The inverse of the pedal of a curve with respect to the same pole is called a *polar reciprocal* of the curve.

For instance, the pedal of a circle with respect to any point O is the limaçon (§ 156)

$$r = a + b\cos\theta;$$

and therefore the polar reciprocal of a circle is the inverse of a limaçon, and its equation is

$$r = c^2/(a + b\cos\theta),$$

the polar equation of a conic section, with a focus at the pole, excentricity $e = b/a$, and semi-latus-rectum $l = c^2/a$.

Again, the polar reciprocal of an epicycloid with respect to the centre O is $r\cos m\theta = b$, a Cotes's spiral (§ 164).

With the usual notation of θ, r, ϕ, p, ρ, ... for the original curve, and θ', r', ϕ', ... for the polar reciprocal with respect to the origin, then from §§ 155, 168,

$$\phi + \phi' = \pi,\ r' = c^2/p,\ p' = c^2/r;$$

and
$$\rho\rho' = r\frac{dr}{dp} \cdot r'\frac{dr'}{dp'} = c^2\frac{r^3}{p^3} = c^2 \operatorname{cosec}^3\phi.$$

171. *Orthogonal and Oblique Trajectories.*

A curve cutting at right angles a system of curves is called an *orthogonal trajectory* of the system; and a curve cutting the system at any constant angle γ other than a right angle is called an *oblique trajectory*.

In § 101, it was shown that the orthogonal trajectories of the cycloids described by all the points on the circumference of a wheel are equal cycloids.

Familiar instances of orthogonal trajectories are seen with (i.) horizontal and vertical straight lines, (ii.) straight lines radiating from a centre, and the concentric circles, (iii.) confocal and coaxial parabolas, (iv.) rectangular hyperbolas, the asymptotes of one system being the axes of the other system, (v.) confocal ellipses and hyperbolas; while the oblique trajectories of system (i.) are inclined straight lines, of (ii.) are equiangular spirals, of (iii.) are confocal but not coaxial parabolas, and of (iv.) are concentric, but not coaxial rectangular hyperbolas.

Since any two curves and the corresponding *inverse* curves cut at the same angle, it follows that the inverse of a system of orthogonal or oblique trajectories forms a new system of orthogonal or oblique trajectories.

We have seen (§ 133) that the graphs of the conjugate functions u and v, given by $u+iv=\mathrm{f}(x+iy)$, when u or v is equated to a constant, form an orthogonal system.

Also, since in an oblique trajectory cutting the curves u and v (§ 133) at constant angles γ and $\frac{1}{2}\pi-\gamma$

$$ds_1\cos\gamma+ds_2\sin\gamma=0,$$

and

$$ds_1/du=ds_2/dv=J^{-\frac{1}{2}};$$

therefore

$$du\cos\gamma+dv\sin\gamma=0,$$

or

$$u\cos\gamma+v\sin\gamma=\text{a constant},$$

is the general equation of an oblique trajectory.

Mr. Larmor has shown (*Proc. London Math. Soc.*, vol. xv.) that in a field of force (§ 83), of which the potential is J, a particle will describe an oblique trajectory with velocity V, if started so that $\frac{1}{2}V^2=J$; or that a ray of light will describe this oblique trajectory if the refractive index varies as $J^{-\frac{1}{2}}$.

Then, if R denotes the radius of curvature of the trajectory, resolving normally,

$$\frac{V^2}{R}=\frac{2J}{R}=\frac{\partial J}{\partial s_2}\cos\gamma+\frac{\partial J}{\partial s_1}\sin\gamma=J^{\frac{1}{2}}\left(\frac{\partial J}{\partial u}\cos\gamma+\frac{\partial J}{\partial v}\sin\gamma\right),$$

or

$$\frac{1}{R}=\frac{\partial J^{\frac{1}{2}}}{\partial u}\cos\gamma+\frac{\partial J^{\frac{1}{2}}}{\partial v}\sin\gamma.$$

Any small contour in the $(x,\ y)$ diagram will be transformed into a similar contour in the $(u,\ v)$ diagram, since angles are unaltered in this transformation, $J^{\frac{1}{2}}$ denoting the scale of the transformation, or the ratio of the linear dimensions of corresponding elements; this is the condition to be satisfied in maps and charts.

Thus, for instance, if

$$u+iv=(x+iy)^n=r^n\cos n\theta+ir^n\sin n\theta,$$

by De Moivre's Theorem (§ 111), then, on putting

$$u+iv=c^{n-1}(x'+iy'),$$

$$u=c^{n-1}x'=r^n\cos n\theta,\ v=c^{n-1}y'=r^n\sin n\theta;$$

and a new system of curves is obtained in which r is changed into r^n/c^{n-1} and θ into $n\theta$; so that we may put

$$r'=r^n/c^{n-1},\ \theta'=n\theta;$$

and now the radial angle ϕ is the same in the old and the new system; and

$$n\frac{r'}{\rho'}-\frac{r}{\rho}=n\frac{dr'\sin\phi}{dr'}-\frac{dr\sin\phi}{dr}=(n-1)\sin\phi.$$

The case of $n=-1$ is equivalent to the operation of *inversion* (§ 168) combined with *reflexion* in Ox.

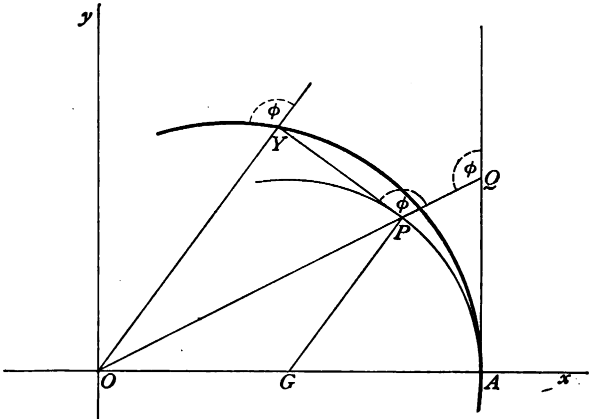


Fig.53

172. *Critical Orbits.*

The orthogonal trajectories given by

$$u = r^n \cos n\theta,\ v = r^n \sin n\theta,$$

are of important interest; the oblique trajectories being

$$u \cos \gamma + v \sin \gamma = r^n \cos(n\theta - \gamma) = \text{a constant}.$$

These curves are called *Critical Orbits* by Clifford (*Kinematic*, p. 113) because they are described as orbits under a central force varying as some power, $2n-3$, of r (§ 84), while the velocity varies as some other power, $n-1$, of r. They are also *catenaries*, the curves assumed by a uniform chain under a central force varying as the power $n-2$ of r.

Writing the equations,

$$r^n \cos n\theta = a^n,\ r^n \sin n\theta = b^n,\ r^n \cos(n\theta - \gamma) = c^n;$$

then (i.) $n = \pm 1$ gives a system of straight lines and circles, (ii.) $n = \pm\frac{1}{2}$ a system of confocal parabolas and cardioids, (iii.) $n = \pm 2$ a system of rectangular hyperbolas and *lemniscates*.

Changing n into $-n$ gives the inverse system of curves

$$r^n = a^n \cos n\theta,\ r^n = b^n \sin n\theta,\ r^n = c^n \cos(n\theta - \gamma);$$

and now, in $r^n = a^n \cos n\theta$,

$$\cot\phi = d\log r/d\theta = -\tan n\theta = \cot(\tfrac{1}{2}\pi + n\theta),$$

or $\phi = \frac{1}{2}\pi + \frac{1}{2}n\theta$; so that (fig. 53)

$$POY = n\theta, \text{ and } AOY = \omega = (n+1)\theta.$$

Then $p = r\cos n\theta = r^{n+1}/a^n$,

or $$p = a(\cos n\theta)^{1+1/n} = a\left(\cos\frac{n\omega}{n+1}\right)^{\frac{n+1}{n}},$$

$$p^m = a^m \cos m\omega, \text{ where } m = n/(n+1);$$

so that the pedal curve AY of the critical orbit AP is also a critical orbit; and the equation of the polar reciprocal of AP may be written

$$r^m \cos m\theta = a^m,$$

another critical orbit.

Consider the path of a ray of light in the Earth's atmosphere, on Simpson's assumption that the refraction index μ varies inversely as the $n+1^{\text{th}}$ power of the Tnuce r from the centre of the Earth.

$u\overset{-}{+}\beta)\mu \sin\phi$ or μp is constant along the ray, or by De Moivre's Theorem the path of the ray is either the

$u + iv = c^{n-}$ an oblique trajectory.

$u = c^{n-1}x' = r^n \cos n\theta$, $v = c$d of curvature through O

and a new system of curves is obt 1),

changed into r^n/c^{n-1} and θ into $n\theta$; s 1)

$$r' = r^n/c^{n-1},\ \theta' = n\theta;$$

d now the radial angle ϕ is the same in pole P of the system; and he straight

$$\frac{r'}{\rho'} - \frac{r}{\rho} = n\frac{dr'\sin\phi}{dr'} - \frac{dr\sin\phi}{dr} = (n-1)\sin\phi.$$ P by y,

of $n = -1$ is equivalent to the operation of 168) combined with *reflexion* in Ox.

$n+1$, so that

$r^n = a^n \cos n\theta$,

ese curves, the chord

$= 2pdr/dp = 2r(n$

$\rho = P_q(n+$

of Critical Orbits.

48) is the roulette of

$a^n \cos n\theta$ with respect

g the angle μ

$m\omega$.

Differentiating logarithmically with respect to s,

$$\frac{m}{y}\frac{dy}{ds} = -\tan\psi\frac{d\psi}{ds};$$

and since $$\frac{dy}{ds} = -\sin\psi,\ \frac{ds}{d\psi} = \rho,$$

therefore $$\rho = y\sec\psi/m = PI/m,$$

so that ρ the radius of curvature of the roulette is $1/m$ or $1+1/n$ times the length of the normal PI, and therefore $m/(m+1)$ times the radius of curvature of the rolling pedal $A'P$; and the evolute of the roulette will possess a similar property.

For instance, if $n=1$, the rolling curve is a circle, and the roulette is a cycloid; and then $\rho = 2PI$ (§ 101), and the evolute of AP is an equal cycloid.

As applications of these roulettes AP, we may instance

(i.) that AP is a catenary curve with Ox horizontal, when the line density is made proportional to the $2+1/n^{\text{th}}$ power of $\sec\psi$, or of the tension; reducing, for $n=-\frac{1}{2}$, to the ordinary catenary, which is therefore the roulette of the focus of a parabola;

(ii.) that, with m negative and $=-p$, AP can be described as a trajectory by a projectile in a resisting medium, in which the retardation due to the resistance is

$$g\left(1-\frac{1}{2p}\right)\left\{1-\left(\frac{v_0}{v}\right)^{4p}\right\}^{\frac{1}{2}},$$

where v denotes the velocity at P, and v_0 at A; reducing, for $p=\frac{1}{2}$, to the ordinary parabolic trajectory, the roulette of the pole of $r^{\frac{1}{3}}\cos\frac{1}{3}\theta = a^{\frac{1}{3}}$, the first negative pedal of the parabola $r^{\frac{1}{2}}\cos\frac{1}{2}\theta = a^{\frac{1}{2}}$;

(iii.) that AP is also the path of a ray of light, when the refractive index varies as y^p; a catenary for $p=1$.

174. *Vectors and Vectorial Equations.*

The quantity $z=x+iy$ is now called the *vector* or *step* from O to the point (x, y); being composed of a step of length x parallel to Ox and then a step of length y parallel to Oy, according to Argand's representation.

Then if $z'=x'+iy'$ represents another vector,

$$z-z'=x-x'+i\,.\,\overline{y-y'}$$

represents the vector or step from (x', y') to (x, y).

It is convenient in physical application to consider the logarithm of the vector, and now

$$w=\log(z-z')$$

or $u+iv=\log(x-x'+i\,.\,\overline{y-y'})$

$$=\log\surd\{(x-x')^2+(y-y')^2\}+i\tan\{(y-y')/(x-x')\}$$

gives u the velocity function, and v the stream function at (x, y) of a *source* or *electrode* at (x', y'); and we may call w the *vector potential.*

Suppose for instance we place n equal positive electrodes at points $A_0, A_1 \dots A_{n-1}$, equally spaced on a circle of radius a; the vector of the point A_s being

$$a\cos 2s\pi/n+ia\sin 2s\pi/n=a\exp(2is\pi/n),$$

the vector potential

$$w=\log\prod_{s=0}^{s=n-1}\{z-a\exp(2is\pi/n)\}=\log(z^n-a^n);$$

and now, with polar coordinates,

$$z\ =r\cos\theta+ir\sin\theta=re^{i\theta},$$

$$z^n=r^ne^{in\theta}=r^n\cos n\theta+ir^n\sin n\theta;$$

$$w=\log(r^n\cos n\theta-a^n+ir^n\sin n\theta),$$

$$u+iv=\log\surd(r^{2n}-2a^nr^n\cos n\theta+a^n)$$
$$+i\tan^{-1}\{r^n\sin n\theta/(r^n\cos n\theta-a^n)\};$$

or $$r^{2n}-2a^nr^n\cos n\theta+a^n=e^{2u},$$

$$r^n\sin(v-n\theta)=a^n\sin v,$$

representing an orthogonal system of curves.

Denoting by r_s, θ_s the polar coordinates of a point P relatively to the origin A_s, then

$$u = \log r_0 r_1 r_2 \dots r_{n-1},\quad v = \theta_0 + \theta_1 + \theta_2 + \dots + \theta_{n-1},$$

giving De Moivre's and Cotes's properties of the circle.

Suppose for instance $n=2$, then

$$u = \log r_1 r_2,\ \text{or}\ r^4 - 2a^2r^2\cos 2\theta + a^4 = e^{2u};$$

and $$v = \theta_0 + \theta_1,\ \text{or}\ r^2\sin(2\theta - v) = -a^2 \sin v,$$

representing *Cassinians*, and the orthogonal rectangular hyperbolas, passing through A_0 and A_1.

When n equal negative electrodes are placed on the circle, at B_1, B_2, ... B_n, midway between the positive electrodes, the vector of B_s being

$$a\exp\{(2s-1)\pi/n\},$$

then their vector potential is

$$\log \prod_{s=1}^{s=n} \{z - a\exp(2s-1)\pi/n\} = \log(z^n + a^n);$$

so that for the system of positive and negative electrodes

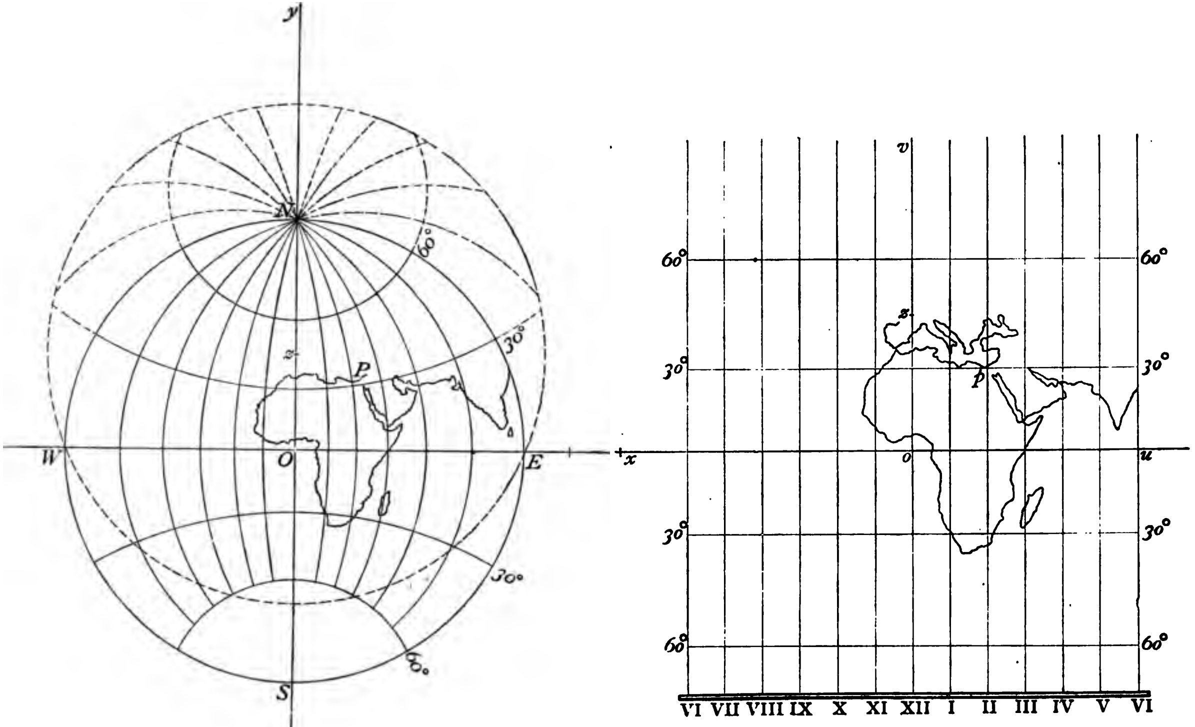
$$\begin{aligned} w &= \log(z^n - a^n)/(z^n + a^n) \\ &= \tfrac{1}{2}\log\frac{r^{2n} - 2a^n r^n \cos n\theta + a^{2n}}{r^{2n} + 2a^n r^n \cos n\theta + a^{2n}} \\ &\qquad + i\tan^{-1}\frac{r^n \sin n\theta}{r^n\cos n\theta - a^n} - i\tan^{-1}\frac{r^n\sin n\theta}{r^n\cos n\theta + a^n} \\ &= -\tanh^{-1}\frac{2a^n r^n\cos n\theta}{r^{2n} + a^{2n}} - i\tan^{-1}\frac{2a^n r^n \sin n\theta}{r^{2n} - a^{2n}}, \end{aligned}$$

giving another orthogonal system of curves,

$$r^{2n} + 2a^n r^n \cos n\theta \coth u + a^{2n} = 0,$$

$$r^{2n} + 2a^n r^n \sin n\theta \quad \cot v - a^{2n} = 0.$$

When $n=1$, these curves are circles, the *dipolar circles* of the next article.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60°
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VI
VII
VIII
IX
X
XI
XII
I
II
III
IV
V
VI

(ii)

Fig. 54

175. *Dipolar Circles.*

When we put $x+iy=c\tan\frac{1}{2}(u+iv)$, and write it

$$x+iy=c\,\frac{\sin\frac{1}{2}(u+iv)\cos\frac{1}{2}(u-iv)}{\cos\frac{1}{2}(u+iv)\cos\frac{1}{2}(u-iv)}=c\,\frac{\sin u+i\sinh v}{\cos u+\ \cosh v},$$

then $$x=\frac{c\sin u}{\cosh v+\cos u},\ y=\frac{c\sinh v}{\cosh v+\cos u},$$

thus dividing $\tan\frac{1}{2}(u+iv)$ into a real and imaginary part.

Also $$\tan u=\tan\{\tfrac{1}{2}(u+iv)+\tfrac{1}{2}(u-iv)\}$$
$$=c\frac{x+iy+x-iy}{c^2-x^2-y^2}=\frac{2cx}{c^2-x^2-y^2},$$

and $$\tan iv=\tan\{\tfrac{1}{2}(u+iv)-\tfrac{1}{2}(u-iv)\}$$
$$=c\frac{x+iy-x+iy}{c^2+x^2+y^2}=\frac{2icy}{c^2+x^2+y^2};$$

so that $$x^2+y^2+2cx\ \cot u-c^2=0\ldots\ldots\ldots\ldots\text{(i.)},$$
$$x^2+y^2-2cy\coth v+c^2=0\ldots\ldots\ldots\ldots\text{(ii.)},$$

the equations of a system of orthogonal dipolar circles (fig. 54.i.); called *dipolar* because the system (i.) represents circles passing through two poles N and S, while system (ii.) represents orthogonal circles having the same radical axis Ox; thus representing the meridians and parallels in the stereographic projection of the terrestrial globe.

Putting u or $v=nt$, we obtain circles described under excentric centres of force; and writing them

$$(x+c\cot u)^2+y^2\ =c^2\text{cosec}^2u\ \ldots\ldots\ldots\ldots\text{(i.)}$$
$$x^2+(y-c\coth v)^2=c^2\text{cosech}^2v\ldots\ldots\ldots\ldots\text{(ii.)},$$

then (i.) represents a circle, centre $(-c\cot u, 0)$, and radius $c\,\text{cosec}\,u$; and (ii.) a circle, centre $(0, c\coth v)$ and radius $c\,\text{cosech}\,v$; and with respect to O, the *power* of (i.) is $-c^2$, and of (ii.) is c^2.

At any point P, the angle $NPS=\pi-u$, while the ratio $SP/NP=e^v$, or $v=\log(SP/NP)$.

If λ denotes the latitude of any parallel of latitude circle (ii.), then since on the stereographic projection the radius of this circle is $c \cot \lambda$, therefore (§ 34)

$$\cot \lambda = \operatorname{cosech} v,\ \tan \lambda = \sinh v,\ \sin \lambda = \tanh v,$$

$$\cos \lambda = \operatorname{sech} v,\ \tan \tfrac{1}{2}\lambda = \tanh \tfrac{1}{2}v, \text{ or } \lambda = \operatorname{gd} v,$$

$$v = \log(\sec \lambda + \tan \lambda) = \log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\lambda).$$

while $\lambda = \operatorname{gd}(u \cot \gamma)$ represents a *loxodrome* or *rhumb line*, cutting the meridians at a constant angle γ.

The inverse system of curves with respect to a pole N or S will be radiating straight lines and concentric circles, representing meridians and parallels in the neighbourhood of a pole, when projected stereographically from the other pole, as in Godfray's Great Circle Chart.

A *great circle* on this chart will be represented by a straight line; and a *rhumb line*, that is an oblique trajectory of the meridians and parallels, will be an equiangular spiral; so that on the stereographic projection it will be an inverse of an equiangular spiral.

The general operation of inversion with respect to any pole, combined with a change of origin, is represented by the *linear substitution* (Schwarz)

$$z' = (az + b)/(Az + B),$$

by which a polygon in the z plane, bounded by straight lines or circles, is transformed into a polygon bounded by circles, cutting at the same angles, in the z' plane.

In Godfray's Great Circle Chart we transform to

$$z' = x' + iy' = c \exp(i\alpha - \beta) = ce^{-\beta}(\cos \alpha + i \sin \alpha),$$

so that α is the longitude, and the latitude λ is given by

$$r' = ce^{-\beta} = c \tan(\tfrac{1}{4}\pi - \tfrac{1}{2}\lambda),$$

or $$\beta = \log(\sec \lambda + \tan \lambda),\ \lambda = \operatorname{gd} \beta.$$

176. *Mercator's Chart.*

Draw any contour on the (u, v) diagram (fig. 54, ii.) corresponding to a contour, the outline of a country, in the (x, y) stereographic diagram; we thus obtain a new map, called Mercator's projection, in which the meridians and parallels are orthogonal systems of parallel straight lines; and while a length u represents the longitude, a length $v=\log(\sec\lambda+\tan\lambda)$ will represent the latitude λ; also $dv/d\lambda=\sec\lambda$, so that the minute of latitude in Mercator's chart increases as the secant of the latitude, being equal to the minute of longitude only at the equator.

Twenty-four standard meridians at equal intervals of 15° in longitude from Greenwich mark the *standard time* at a place; the standard time being the mean solar time at the nearest standard meridian.

Taking the Godfray Chart as representing any system of parallels and meridians with respect to an axis through any zenith Z, say in longitude 0 and latitude $\operatorname{gd} b$, then the representation on Mercator's Chart will be given by

$$\tan\tfrac{1}{2}(u+i\,.\,v-b)=\exp(ia-\beta)=x'+iy'=z';$$

or, dropping b as representing a mere change of origin,

$$ia-\beta=\log\tan\tfrac{1}{2}(u+iv)$$

$$=\log\frac{\sin u+i\sinh v}{\cos u+\cosh v}=i\tan^{-1}\frac{\sinh v}{\sin u}-\coth^{-1}\frac{\cosh v}{\cos u};$$

so that $\sinh v=\tan a\sin u,\ \cosh v=\coth\beta\cos u$;

the representation on Mercator's Chart of a system of parallel small circles and their meridians; these lines are called *Sumner lines* (p. 203), being the lines on Mercator's Chart on which a celestial body in the zenith at Z will at any instant have the same altitude.

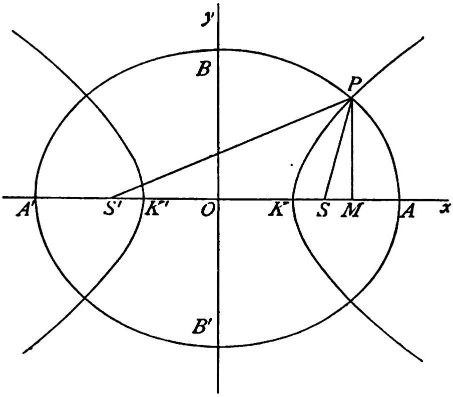


Fig. 55

177. *Confocal Ellipses and Hyperbolas.*

As another practical illustration of conjugate functions, put

$$x+iy=c\cosh(u+iv)$$
$$=c(\cosh u\cos v+i\sinh u\sin v);$$

then $x=c\cosh u\cos v,\ y=c\sinh u\sin v$;

and alternately eliminating u and v,

$$\frac{x^2}{c^2\cosh^2 u}+\frac{y^2}{c^2\sinh^2 u}=1 \text{.................(i.)}$$

$$\frac{x^2}{c^2\cos^2 v}-\frac{y^2}{c^2\sin^2 v}=1 \text{................(ii.)}$$

representing a system of confocal ellipses and hyperbolas for constant values of u and v the polar reciprocals with respect to N or S of the circles v of § 175; and sech u will be the excentricity of the ellipse, sec v of the hyperbola (fig. 55); while (§ 35) v will be the excentric angle or excentric anomaly of a point P on the ellipse AP, and u will be hyperbolic excentric anomaly of the point P on the hyperbola KP.

Put $v=nt$, then

$$x+iy=c\cosh(u+int)=c\cos(nt-iu)$$

represents confocal ellipses described under a central attraction to O, varying as the distance, the period being $2\pi/n$; while hyperbolas are described with $u=nt$, and $x+iy=c\cosh(nt+iv)$.

In an oblique trajectory

$$u\cos\gamma+v\sin\gamma=\text{constant},$$

we may put $v=n(t-\tau)\cos\gamma$, $u=-n(t-\tau')\sin\gamma$;
so that $x+iy=c\cos n\{(t-\tau)\cos\gamma+i(t-\tau')\sin\gamma\}$;
and the equation is obtained by eliminating t between

$$x=c\cos\{n(t-\tau)\cos\gamma\}\cosh\{n(t-\tau')\sin\gamma\},$$
$$y=-c\sin\{n(t-\tau)\cos\gamma\}\sinh\{n(t-\tau')\sin\gamma\}.$$

The equations of the tangent and normal of the ellipse at P, which are also the normal and tangent of the hyperbola at P, are

$$x\frac{\cos v}{\cosh u}+y\frac{\sin v}{\sinh u}=c \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii.)},$$

$$x\frac{\cosh u}{\cos v}-y\frac{\sinh u}{\sin v}=c \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv.)}.$$

Denoting by p and q the lengths of the perpendiculars from O upon the tangents at P of the ellipse and hyperbola,

$$p=c\cosh u\sinh u/(\cosh^2u-\cos^2v)^{\frac{1}{2}},$$
$$q=c\quad\sin v\quad\cos v/(\cosh^2u-\cos^2v)^{\frac{1}{2}};$$

The pole of (iii.) with respect to the hyperbola (ii.) will be $(c\cos^3v/\cosh u,\ -c\sin^3v/\sinh u)$, and this pole will therefore be Q, the centre of curvature of the ellipse at P (§ 98); and $PQ=\rho=c(\cosh^2u-\cos^2v)^{\frac{3}{2}}/(\cosh u\sinh u)$.

Similarly $(c\cosh^3u/\cos v,\ -c\sinh^3u/\sin v)$, the pole of (iv.) with respect to (i.), is the centre of curvature of the hyperbola at P; and the radius of curvature is

$$c(\cosh^2u-\cos^2v)^{\frac{3}{2}}/(\cos v\sin v).$$

We may consider $w=u+iv$ as representing a vector of curvilinear steps u and v; u along a hyperbola and v along a confocal ellipse, meeting at the point

$$z=x+iy=c\cosh(u+iv).$$

If P' is any other point given by

$$z'=x'+iy'=c\cosh(u'+iv'),$$

then the vector $P'P$ is given by

$$\begin{aligned} z-z' &= c\{\cosh(u+iv)-\cosh(u'+iv')\} \\ &= 2c\sinh\tfrac{1}{2}(u+u'+i\,.\,v+v')\sinh\tfrac{1}{2}(u-u'+i\,.\,v-v'); \end{aligned}$$

so that PP'^2

$$\begin{aligned} &= 4c^2\sinh\tfrac{1}{2}(u+u'+i\,.\,v+v')\sinh\tfrac{1}{2}(u-u'+i\,.\,v-v') \\ &\qquad \sinh\tfrac{1}{2}(u+u'-i\,.\,v+v')\sinh\tfrac{1}{2}(u-u'-i\,.\,v-v') \\ &= c^2\{\cosh(u+u')-\cos(v+v')\}\{\cosh(u-u')-\cos(v-v')\}. \end{aligned}$$

This expression is unaltered by an interchange of u, u', or v, v'; so that if the points p, p', called *corresponding points*, are given by (u, v'), (u', v), then $pp'=PP'$.

Further, if ϕ denotes the angle between PP' and pp',

then
$$\begin{aligned} i\phi &= \log\frac{\cosh(u+iv)-\cosh(u'+iv')}{\cosh(u+iv')-\cosh(u'+iv)} \\ &= \log\frac{\tanh\frac{1}{2}(u-u')+i\tan\frac{1}{2}(v-v')}{\tanh\frac{1}{2}(u-u')-i\tan\frac{1}{2}(v-v')} \\ &= 2i\tan^{-1}\frac{\tan\frac{1}{2}(v-v')}{\tanh\frac{1}{2}(u-u')}. \end{aligned}$$

Captain Weir's Azimuth Diagram consists of a series of confocals, the hyperbolas marking the longitude or hour angle v, while the ellipses are marked with the degrees in $\operatorname{gd} u$, to represent latitude or declination.

Now, if the hour angle is the complement of v, and if the latitude and declination are $\operatorname{gd} u$ and $\operatorname{gd} d$, a straight line joining the points (u, v) and $(d, 0)$ will give the azimuth or bearing of the object from the meridian Oy.

(Godfray, *Astronomy*, § 222.)

Changing to the focus S or S' as origin
$$x-c+iy=2c\sinh^2\tfrac{1}{2}(u+iv),\quad x+c+iy=2c\cosh^2\tfrac{1}{2}(u+iv),$$
so that if r, r' denote the focal distances SP, $S'P$,
$$r=2c\sinh\tfrac{1}{2}(u+iv)\sinh\tfrac{1}{2}(u-iv)$$
$$=c(\cosh u-\cos v)=a(1-e\cos v);$$
$$r'=c(\cosh u+\cos v)=a(1+e\cos v).$$
Then $$r'+r=2c\cosh u=AA' \text{ .... (v.)},$$
$$r'-r=2c\cos v=KK' \text{ ....(vi.)};$$
and hence $SP=AK$, $S'P=KA'$; also $OP=KB$.

Denoting the angles ASP, $AS'P$ by θ, θ', then in Astronomy θ is called the *true anomaly* of P, reckoned from *perihelion* at A, the Sun being supposed to be at the focus S; but θ' is the true anomaly from *aphelion* A, with the Sun now at the other focus S'.

Then
$$\cos\theta=\frac{x-c}{r}=\frac{\cosh u\cos v-1}{\cosh u-\cos v},\ \cos\theta'=\frac{\cosh u\cos v+1}{\cosh u+\cos v};$$
or $$\sin\theta=\frac{\sinh u\sin v}{\cosh u-\cos v},\qquad \sin\theta'=\frac{\sinh u\sin v}{\cosh u+\cos v};$$
$$\tan\tfrac{1}{2}\theta=\coth\tfrac{1}{2}u\tan\tfrac{1}{2}v,\ \tan\tfrac{1}{2}\theta'=\tanh\tfrac{1}{2}u\tan\tfrac{1}{2}v;$$
and $$r=SA\cos^2\tfrac{1}{2}v\sec^2\tfrac{1}{2}\theta,\qquad r'=SA'\cos^2\tfrac{1}{2}v\sec^2\tfrac{1}{2}\theta'.$$

Again, since $$\cos v=\frac{1+\cos\theta\cosh u}{\cosh u+\cos\theta}=\frac{e+\cos\theta}{1+e\cos\theta},$$
or $$(1+e\cos\theta)(1-e\cos v)=1-e^2;$$
therefore $$r=c\sinh^2u/(\cosh u+\cos\theta),$$
of the form $r=l/(1+e\cos\theta)$, with $l=c\sinh^2u/\cosh u$, the polar equation of the ellipse, with origin at S.

Similarly $$r'=c\sinh^2u/(\cosh u-\cos\theta')$$
is the polar equation of the ellipse, with origin at S'; while for the hyperbola the corresponding polar equations are $$r=\frac{c\sin^2v}{\cos v-\cos\theta},\ r'=\frac{c\sin^2v}{\cos\theta'-\cos v}.$$

178. *Confocal Limaçons.*

By inversion of this system of conics with respect to either focus, say S, taking $2c$ as the constant of inversion,

$$x+iy=2c\operatorname{cosech}^2\tfrac{1}{2}(u+iv),$$
$$x+2c+iy=2c\coth^2\tfrac{1}{2}(u+iv);$$

and

$$r\sinh^2u=4c(\cosh u+\cos\theta)\;\ldots\ldots\ldots\ldots\;\text{(vii.)},$$
$$r\sin^2v=4c(\cos v-\cos\theta)\;\ldots\ldots\ldots\ldots\;\text{(viii.)},$$

are the inverses of the ellipse (i.) and hyperbola (ii.), and the pedals with respect to N or S of the circles v of § 175, a system of confocal limaçons (§ 155).

Denoting now by r' the distance from the other focus S', the position of which is unaltered by inversion, then

$$r'=2c\coth\tfrac{1}{2}(u+iv)\coth\tfrac{1}{2}(u-iv)=2c\frac{\cosh u+\cos v}{\cosh u-\cos v},$$

while

$$r=2c\operatorname{cosech}\tfrac{1}{2}(u+iv)\operatorname{cosech}\tfrac{1}{2}(u-iv)=\frac{4c}{\cosh u-\cos v};$$

so that

$$r'+2c=\frac{4c\cosh u}{\cosh u-\cos v}=r\cosh u\;\ldots\ldots\ldots\ldots\;\text{(ix.)},$$
$$r'-2c=\frac{4c\cos v}{\cosh u-\cos v}=r\cos v\;\ldots\ldots\ldots\ldots\;\text{(x.)},$$

equations of the form $r'-lr=$ constant, which are *Cartesian ovals* in the general case, the inverse of a conic with respect to any point on the transverse axis.

The limaçon in which $v=\frac{1}{3}\pi$, $l=\frac{1}{2}$, is called the *Trisectrix*.

For if $A'OR$ is any angle to be trisected (fig. 47, ii.), and if $A'R$ is joined cutting the curve in p; then, if $A'Op=\theta$,

$$Op=OA'(2\cos\theta-1);\text{ and }A'p=2OA'\sin\tfrac{1}{2}\theta=A'q,$$

if Op meets the arc $A'R$ in q.

Therefore $\theta=qpA'-OA'p=pqA'-OA'p=OA'q-OA'p=pA'q=\frac{1}{2}qOR$; so that Op is a trisector of the angle $A'OR$.

179. *Kepler's Laws of Planetary Motion.*

Kepler, by long continued observations and measurement of the Sun's diameter and motion in longitude, noticed that

(i.) the variation of the Sun's apparent diameter d could be expressed by the formula $D(1+e\cos\theta)$, where D denotes the mean angular diameter (about 32′) and θ the Sun's longitude from perihelion, e being a small constant, about 1/60;

(ii.) that the Sun's daily motion in longitude was proportional to the apparent area or square of the diameter.

Since the apparent diameter is inversely proportional to the distance, he deduced the laws called Kepler's Laws—

(i.) that the relative orbit of the Earth (or any planet) and of the Sun is an ellipse, with a focus at the Sun, if the Sun is supposed fixed, given in polar coordinates by $l/r=d/D=1+e\cos\theta$.

(ii.) that $r^2 d\theta/dt=h$ is constant, and that the Earth therefore sweeps out by its radius vector from the Sun equal areas in equal times.

The third law of Kepler—(iii.) the squares of the periodic times of the planets round the Sun are proportional to the cubes of their mean distances—was easily inferred by arithmetical calculation, when once the distances of the planets from the Sun were measured, in terms of the Sun's distance from the Earth.

Newton inferred from law (ii.) that the earth is attracted by the Sun; and from law (i.) that the attraction must be inversely proportional to the square of the distance; while law (iii.) showed that the attraction of the Sun was of the same nature on all the planets (§ 84).

By Newton's Law of Universal Gravitation, employing C.G.S. units (§ 142), the attraction between two spherical bodies, for instance, the Sun and the Earth, weighing S and E grammes, when their centres are a centimetres apart, will be given by the expression $CSEa^{-2}$ (*dynes*); and C is called the *constant of gravitation*, being the attraction in dynes between two spheres, each weighing one gramme, when their centres are one centimetre apart.

Then Kepler's Third Law, in a mathematical form, asserts that if T is the period in seconds of the Sun and planet, and if $n=2\pi/T$ denotes the *mean motion*, then by ex. 1, p. 175, $n^2a^3=4\pi^2a^3/T^2=C(S+E)$.

According to the Cavendish experiment, now being repeated with improved apparatus by Mr. C. V. Boys, we may take $C=10^{-8}\times 6{\cdot}48$, $1/C=10^7\times 1{\cdot}54$.

(Everett, *Units and Physical Constants.*)

Denoting by g the acceleration (in C.G.S. *spouds*) of the attraction of the Earth, then

$$g=CE/R^2, \text{ or } CE=gR^2;$$

so that we can determine E when C is known, and *vice versa;* and till this is done, Newton's *Principia* is merely Kinematics.

With the above value of C, and $g=981$, $R=10^9\div\frac{1}{2}\pi$,

$$E=gR^2/C=10^{27}\times 6{\cdot}12 \text{ grammes};$$

giving a mean density $\rho=E/(\frac{4}{3}\pi R^3)=5{\cdot}67$.

Denoting by Π the Sun's parallax ($8''{\cdot}76$), and by T the number of seconds in one year, then the attraction between the Sun and the Earth is

$$F=4\pi^2ER\operatorname{cosec}\Pi/T^2=10^{27}\times 3{\cdot}65 \text{ dynes},$$

while $\quad S=Fa^2/CE=10^{33}\times 1{\cdot}2$ grammes,

with the above numerical values.

180. *Elliptic Planetary Motion.*

Considering an elliptic orbit AP (fig. 55) in which $c\cosh u = OA = a$, $c\sinh u = OB = b$, $c\sinh^2 u/\cosh u = l$, the semi-latus rectum, and $\operatorname{sech} u = e$; then the sector

$$ASP = \text{sector } AOP - \text{triangle } OSP = \tfrac{1}{2}ab(v - e\sin v).$$

If the ellipse is described in period T round the Sun at S, with mean angular velocity $n = 2\pi/T$, and if the time to P from perihelion at A is denoted by t, then by Kepler's Second Law,

$$\frac{t}{T} = \frac{\tfrac{1}{2}ab(v - e\sin v)}{\pi ab},$$

or

$$nt = v - e\sin v;$$

and now nt is called the *mean anomaly*, being the true anomaly of a planet moving uniformly in a circular orbit in the same period T.

Expressed in terms of the true anomaly θ from perihelion A,

$$\cos v = \frac{e + \cos\theta}{1 + e\cos\theta}, \quad \sin v = \frac{\sqrt{(1 - e^2)}\sin\theta}{1 + e\cos\theta},$$

and

$$nt = \sin^{-1}\frac{\sqrt{(1 - e^2)}\sin\theta}{1 + e\cos\theta} - \frac{e\sqrt{(1 - e^2)}\sin\theta}{1 + e\cos\theta}.$$

Reckoned from aphelion A with the Sun at S',

$$nt' = v + e\sin v$$
$$= \sin^{-1}\frac{\sqrt{(1 - e^2)}\sin\theta'}{1 - e\cos\theta'} + \frac{e\sqrt{(1 - e^2)}\sin\theta'}{1 - e\cos\theta'}.$$

For completeness we may consider the hyperbolic branch KP, of excentricity e', described round the Sun at S, and now the mean anomaly

$$n't = e'\sinh u - u$$
$$= \frac{e'\sqrt{(e'^2 - 1)}\sin\theta}{1 + e'\cos\theta} - \sinh^{-1}\frac{\sqrt{(e'^2 - 1)}\sin\theta}{1 + e'\cos\theta},$$

with

$$n'a'^3 = C(S + E),$$

a' denoting the semi-transverse axis OK, and the true anomaly θ being now KSP, reckoned from perihelion K.

With the Sun at S', the hyperbolic branch KP will be described under a repulsion from S', and

$$n't' = e'\sinh u + u$$
$$= \frac{e'\sqrt{(e'^2-1)}\sin\theta'}{e'\cos\theta'-1} + \sinh^{-1}\frac{\sqrt{(e'^2-1)}\sin\theta'}{e'\cos\theta'-1}.$$

181. *Rectilinear Motion under Varying Gravity.*

By making $e=1$ or $u=0$, the ellipse becomes a straight line, and we obtain the solution for rectilinear motion under attraction to a fixed centre, varying inversely as the square of the distance; for instance, the motion of a body shot vertically upwards from the Earth with very great velocity.

Then, if R denotes the radius of the Earth,

$$\frac{d^2x}{dt^2} = -g\frac{R^2}{x^2}, \text{ and } \frac{1}{2}\frac{dx^2}{dt^2} = gR^2\left(\frac{1}{x}-\frac{1}{2a}\right),$$

if the body reaches a distance $2a$ from the centre of the Earth; and if E denotes the weight of the Earth, and C the gravitation constant, $n^2a^3 = gR^2 = CE$.

Now $\quad x = a(1+\cos v)$, where $nt = v + \sin v$,
for an attraction; but for a repulsion

$$x = a(\cosh u + 1), \text{ where } nt = \sinh u + u.$$

If O denotes the centre of the Earth and the body starting from A falls to P in the time t, and if PQ is drawn at right angles to OA to meet the circle described on OA as diameter in Q; then the angle $AOQ = \frac{1}{2}v$, and the area $\quad AOQ = \frac{1}{4}a^2(v+\sin v) = \frac{1}{4}a^2nt$;
so that the time from A to P is proportional to the area AOQ, and the point Q moves freely on the circle AQ, under an attraction to O.

To shoot a body weighing W lb. vertically upwards away from the Earth to a distance $2a$ feet from the centre, say the distance of the Moon, requires energy, in ft. lb.,

$$\tfrac{1}{2}Wv^2/g = WR^2\left(\frac{1}{R}-\frac{1}{2a}\right) = WR\left(1-\frac{R}{2a}\right);$$

and with the Moon's parallax of 57′, $2a/R = \operatorname{cosec} 57' = 60$, about; while R, the radius of the Earth in feet (§ 99),

$$= 90 \times 60 \times 6080 \div \tfrac{1}{2}\pi.$$

If we could shoot a body up to a height R at the North Pole, then fired horizontally the body would, in the absence of resistance, skim the surface of the Earth like a grazing satellite, in a period $2\pi/n = 2\pi\sqrt{(R/g)}$ seconds; and $n^2 = g/R = CE/R^3 = \frac{4}{3}\pi\rho C$.

Assuming that gravity inside the Earth varies as the distance from the centre, as would be the case if the Earth were homogeneous, $2\pi\sqrt{(R/g)}$ sec. is also the period of oscillation in a diametral tunnel from pole to pole.

The period of the Moon being about 27 days, the period of a satellite about one-ninth the distance of the Moon would be, by Kepler's Third Law, about one day, and the satellite would appear stationary in the sky; and this would make the period of a grazing satellite $27 \div (60)^{\frac{3}{2}}$ or about one 17th of a day.

In the plane of the equator such a satellite would fly past a point 16 or 18 times a day, every 90 or 80 minutes, according as it moved eastward or westward; and the velocity over the ground would be $7\frac{11}{27}$ or $8\frac{1}{3}$ kilometres per second.

If the Earth was made to rotate 17 times faster, without change of shape, bodies at the equator would be on the point of flying off into space, and elsewhere the plumb line would point to the Pole Star.

182. *Kepler's Problem.*

This is the problem in Astronomy, to express the true anomaly θ and the excentric anomaly v in terms of the mean anomaly nt; and for this purpose we acquire the series of Lagrange (§ 150) and Fourier (§ 183).

Denoting by m the mean anomaly nt, or more strictly, in astronomical notation, $nt+\epsilon-\varpi$, where ϵ denotes the *epoch*, and ϖ the *longitude of perihelion*, then

$$v=m+e\sin v,$$

which, by Lagrange's Theorem (§ 150), gives in a series proceeding in ascending powers of e,

$$v=\quad m+\Sigma\ \frac{e^p}{p!}\ \frac{d^{p-1}}{dm^{p-1}}(\sin m)^p\,;$$

$$\sin v=\sin m+\Sigma\frac{e^p}{(p+1)!}\frac{d^p}{dm^p}\ (\sin m)^{p+1}.$$

Similarly $\sin 2v$, $\sin 3v$, ..., $\sin pv$, and $\cos v$ can be expanded in powers of e; and

$$\frac{r}{a}=1-e\cos v=\frac{dm}{dv}\,;$$

$$\frac{a}{r}=\frac{dv}{dm}=1+\Sigma\frac{e^p}{p!}\frac{d^p}{dm^p}(\sin m)^p\,;$$

while from $\tan\frac{1}{2}\theta=\coth\frac{1}{2}u\tan\frac{1}{2}v$,

we deduce by logarithmic differentiation

$$\frac{d\theta}{\sin\theta}=\frac{dv}{\sin v},\text{ or }\frac{d\theta}{dv}=\frac{\sinh u}{\cosh u-\cos v}=\frac{1-c^2}{1-2c\cos v+c^2},$$

with $c=e^{-u}$; and, resolved into partial fractions,

$$\frac{d\theta}{dv}=\frac{1}{1-ce^{iv}}+\frac{1}{1-ce^{-iv}}-1$$

$$=1+2\Sigma c^p\cos pv,$$

$$\theta=v+2\Sigma\frac{c^p}{p}\sin pv,$$

which can now be expressed in terms of m.

We can rearrange these series so as to proceed according to cosines or sines of multiples of the mean anomaly *m*, the coefficients being functions of *e*, and this is more convenient for Kepler's Problem.

A series of this nature is called a *Fourier Series*, and we proceed to show how the coefficients of such a series can be calculated, for any arbitrary single valued function fx, in a series proceeding by cosines or sines of multiples of x/l, when l is any arbitrary quantity.

183. *Fourier's Series.*

Assume that, between the limits $x = \pm l$, the function fx can be expressed by a series of the form

$$\text{f}x = \tfrac{1}{2}A_0 + \Sigma A_p \cos(p\pi x/l) + \Sigma B_p \sin(p\pi x/l),$$

where Σ denotes the sum of the series obtained by giving p all positive integral values from 1 to ∞; it is required to determine the A's and B's.

Divide the function fx into its *odd* and *even* part (§ 46), and denote them by f_1x and f_2x respectively; thus

$$\text{f}_2x = \tfrac{1}{2}\{\text{f}(x) + \text{f}(-x)\} = \tfrac{1}{2}A_0 + \Sigma A_p \cos(p\pi x/l) \ldots\ldots\text{(i.)},$$

$$\text{f}_1x = \tfrac{1}{2}\{\text{f}(x) - \text{f}(-x)\} = \qquad \Sigma B_p \sin(p\pi x/l) \ldots\ldots\text{(ii.)}.$$

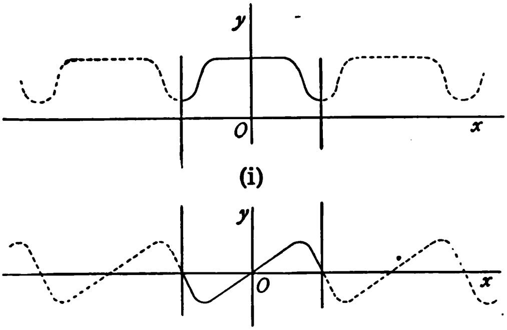
To determine A_p, change x into v in (i.), multiply both sides by $\cos(p\pi v/l)$, and integrate with respect to v between the limits 0 and l; then since

$$\int_0^l \cos\frac{p\pi v}{l}\cos\frac{q\pi v}{l}dv = \int_0^l \left\{\tfrac{1}{2}\cos(p-q)\frac{\pi v}{l} + \tfrac{1}{2}\cos(p+q)\frac{\pi v}{l}\right\}dv$$

$$= \left[\frac{\sin\{(p-q)\pi v/l\}}{2(p-q)\pi/l} + \frac{\sin\{(p+q)\pi v/l\}}{2(p+q)\pi/l}\right]_0^l = 0;$$

while $$\int_0^l \cos^2\frac{p\pi v}{l}dv = \int_0^l \left(\tfrac{1}{2} + \tfrac{1}{2}\cos\frac{2p\pi v}{l}\right)dv = \tfrac{1}{2}l;$$

therefore $$\int_0^l \text{f}_2v\cos(p\pi v/l)dv = \tfrac{1}{2}lA_p;\ \int_0^l \text{f}_2v\,dv = \tfrac{1}{2}lA_0.$$



(i)

(ii)

Fig. 56

Similarly, changing x into v in (ii.), multiplying by $\sin(p\pi v/l)$, and integrating with respect to v between 0 and l, we find

$$\int_0^l \mathrm{f}_1 v \sin(p\pi v/l)dv = \tfrac{1}{2}lB_p.$$

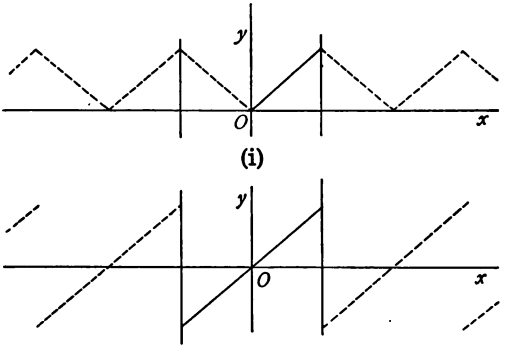
Therefore, between the limits $x = \pm l$,

$$\mathrm{f}x = \frac{1}{l}\int_0^l \mathrm{f}_2 v dv + \frac{2}{l}\Sigma \cos\frac{p\pi x}{l}\int_0^l \mathrm{f}_2 v \cos\frac{p\pi v}{l}dv$$
$$+\frac{2}{l}\Sigma \sin\frac{p\pi x}{l}\int_0^l \mathrm{f}_1 v \sin\frac{p\pi v}{l}dv,$$

and this is called *Fourier's Series.*

By supposing p to take all integral values, positive or negative, we may write Fourier's series,

$$\mathrm{f}x = \frac{1}{2l}\sum_{p=-\infty}^{p=\infty}\int_{-l}^{l} \mathrm{f}v \cos\frac{p\pi(x-v)}{l}dv.$$



(i)

(ii)

Fig. 57

At the limits $x = \pm l$, the value of the series is $\mathrm{f}_2 l$, so that in general there is discontinuity at the limits; and outside these limits Fourier's series represents periodic repetitions of the function fx between the limits.

This is exhibited by drawing the graphs of

(i.) $\frac{1}{2}A_0 + \Sigma A_p \cos(p\pi x/l)$ and (ii.) $\Sigma B_p \sin(p\pi x/l)$,

as exhibited in fig. 56, (i.) and (ii.)

When the limits between which fx is to be expressed by Fourier's Series are 0 and l, then either series (i.) or (ii.) may be chosen at pleasure, to represent fx, but it is best to choose the series which introduces the least discontinuity at the limits.

For instance, suppose f$x = x$; then between 0 and l,

$$\text{(i.)}\quad x = l - \frac{l}{(\frac{1}{2}\pi)^2}\Sigma\frac{1}{(2p-1)^2}\cos(2p-1)\frac{\pi x}{l};$$

$$\text{or}\qquad \text{(ii.)}\quad x = \frac{l}{\frac{1}{2}\pi}\Sigma\frac{(-1)^{p-1}}{p}\sin\frac{p\pi x}{l};$$

but outside these limits the series (i.) and (ii.) represent the dotted lines in fig. 57, (i.) and (ii.).

The student is recommended to draw the graphs of the first two or three terms in a Fourier series, to see how quickly the series approximates to the given function.

For instance, draw the graphs of

$$\sin x - \tfrac{1}{2}\sin 2x,\ \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x,\ \ldots\ldots$$

$$\cos x + \tfrac{1}{9}\cos 3x,\ \cos x + \tfrac{1}{9}\cos 3x + \tfrac{1}{25}\cos 5x,\ \ldots\ldots$$

Fourier made great use of his series in problems on the Conduction of Heat, where a solution of the partial differential equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ is required, giving the temperature u at any time t; for if the initial state of temperature U is expressed by the Fourier Series

$$U = \tfrac{1}{2}A_0 + \Sigma(A_p\cos px + B_p\sin px),$$

then at any subsequent time t the temperature

$$u = \tfrac{1}{2}A_0 + \Sigma(A_p\cos px + B_p\sin px)e^{-p^2kt}.$$

The geometrical meaning of the coefficients A_p and B_p is explained by Clifford (*Proc. London Math. Soc.*, vol. v.) according to the method employed in Tidal Harmonic Analysis (G. H. Darwin, *Brit. Ass. Report,* 1883), where a pencil attached to a float registers fx, the rise and fall of the tide, on a cylinder turning uniformly by clockwork round a vertical axis.

To analyse the tides of period P, the cylinder is made to revolve in a period P/p, when a closed curve will be traced on the cylinder after p revolutions.

The plane on which the projection of this closed curve has a maximum area A, with attention to the sign of the area (§ 137), will cut the cylinder in an ellipse of area A/p.

Now, if the pencil is made to follow this ellipse as the cylinder revolves in the period P/p, the pencil will have the component harmonic travel

$$A_p\cos(p\pi x/l) + B_p\sin(p\pi x/l).$$

184. *Fourier's Series in Kepler's Problem.*

Given the relation connecting the mean and excentric anomalies, $v=m+e\sin v$,

then $$\frac{dv}{dm}=\frac{1}{1-e\cos v}=\frac{1+e\cos\theta}{1-e^2}=\frac{a}{r}=1+\Sigma A_p\cos pm,$$

suppose, when expressed by a Fourier series; and now

$$\tfrac{1}{2}A_p=\frac{1}{\pi}\int_0^\pi\frac{dv}{dm}\cos pm\,dm=\frac{1}{\pi}\int_0^\pi\cos p(v-e\sin v)dv;$$

and this definite integral is a function of pe, called *Bessel's function* of the order p, and denoted by $J_p(pe)$; and it can be verified, as in § 207, that $v=J_m(qr)$ satisfies Bessel's differential equation on p. 185 (§ 88).

Now $$\frac{a}{r}=\frac{dv}{dm}=1+2\Sigma J_p(pe)\cos pm;$$

and integrating with respect to m,

$$v=m+2\Sigma\frac{J_p(pe)}{p}\sin pm,$$

giving the excentric anomaly v in terms of the mean anomaly m. Again

$$\sin v\frac{dv}{dm}=\frac{dv}{de}=2\Sigma J'_p(pe)\sin pm;$$

and integrating with respect to m,

$$\cos v=C+2\Sigma\frac{J'_p(pe)}{p}\cos pm,$$

where

$$\pi C=\int_0^\pi\cos v\,dm=\int_0^\pi\cos v(1-e\cos v)dv=-\tfrac{1}{2}\pi e,\quad C=-\tfrac{1}{2}e;$$

and $$\sin v=\frac{v-m}{e}=2\Sigma\frac{J_p(pe)}{pe}\sin pm;$$

whence $x=a\cos v$, $y=b\sin v$, and $r=a(1-e\cos v)$ can be expressed by a Fourier series in terms of the mean anomaly m.

Examples on Fourier Series.

(1) Prove that, in a series of cosines, between $\pm\tfrac12 l$,

$$x^2=\tfrac14 l^2+\frac{l^2}{(\frac12\pi)^3}\Sigma\frac{(-1)^p}{(2p-1)^3}\cos(2p-1)\frac{\pi x}{l}.$$

Prove that, between 0 and π,

$$\frac{\sin x}{1^3}+\frac{\sin 3x}{3^3}+\frac{\sin 5x}{5^3}+\dots \text{ or } \Sigma\frac{\sin(2p-1)x}{(2p-1)^3}=\tfrac18\pi^2x(\pi-x);$$

and give the value of the series outside these limits.

(2) In a series of cosines,

$$\cosh mx=2\pi\sinh ml\left\{\frac{1}{2m^2l^2}+\Sigma\frac{(-1)^p}{m^2l^2+p^2\pi^2}\cos\frac{p\pi x}{l}\right\}.$$

(3) In a series of sines,

$$\sinh mx=2\pi\sinh ml\Sigma\frac{(-1)^{p-1}}{m^2l^2+p^2\pi^2}p\sin\frac{p\pi x}{l}.$$

Deduce the expansions of e^{mx} and e^{-mx}.

(4) Expand $\cos mx$ in a series of cosines, and $\sin mx$ in a series of sines, of multiples of $\pi x/l$.

(5) Show that the Fourier series which represents the density or temperature in an endless wire must in general contain both sines and cosines.

Determine the Fourier series to represent the temperature of a circular wire, in which the temperatures of the four quadrants in order are 1, 2, 3, 4.

(6) Prove that the equation

$$y=\tfrac12 l+x-\frac{l}{(\frac12\pi)^2}\Sigma\frac{1}{(2p-1)^2}\cos(2p-1)\frac{\pi(x+y)}{l}$$

represents a staircase, of vertical and horizontal steps of length l.

(7) Prove that the equation

$$\frac{\pi^2}{24}=\Sigma\frac{(-1)^p}{p^2}\cos\tfrac12 p(x+y)\cos\tfrac12 p(x-y)$$

represents circles of radius π; and draw them.

CHAPTER VII.

INTEGRATION IN GENERAL.

185. In the preceding chapters a sketch of Integration has been given, and then a number of applications, intended to show the practical use of the method; and now it is proposed to resume the consideration of Integration from a more general and systematic standpoint, so that the student may more readily perceive and write down the required result.

The process of Integration is necessarily of a tentative nature, depending on a previous knowledge of Differentiation; and in general the most convenient order of the mental operations employed in the integration of a given function will be found to be:

(i.) to guess the function required for the integration;
(ii.) to assign the argument of this function;
(iii.) to write down the proper constant or numerical factors of the integrated function.

Of these three operations, the first is of the most fundamental importance, depending as it does on the principles of the Calculus, but it is the second operation which presents the greatest practical difficulty, while the third only requires verification by a mental differentiation.

186. *General Integration of Algebraical Functions.*

The most general algebraical function of x which is capable of integration by means of the preceding stock of functions can be written

$$\frac{S+T\sqrt{R}}{U+V\sqrt{R}},$$

where S, T, U, V are rational integral algebraical functions of x, and R is a linear or quadratic function of x, and therefore of the form $ax^2+2bx+c$.

If R is of the third or fourth degree in x, *elliptic functions* are in general required for the integration; and if R is of the fifth or higher degree, *hyperelliptic functions* are required.

We first rationalize the denominator, when

$$\frac{S+T\sqrt{R}}{U+V\sqrt{R}}=\frac{(S+T\sqrt{R})(U-V\sqrt{R})}{U^2-V^2R}=\frac{M}{D}+\frac{N}{D}\frac{1}{\sqrt{R}},$$

where $D=U^2-V^2R$, $M=SU-TVR$, $N=(TU-SV)R$;

and D, M, N are thus rational integral functions of x.

To integrate the rational function M/D, this function is split up into its quotient and partial fractions, in the manner explained in treatises on Algebra; and then the integration of each term is in general easily effected.

To integrate the irrational part, $N/D\sqrt{R}$, we may resolve the rational function N/D into its partial fractions, and integrate each term by appropriate substitutions.

187. *Integration of Rational Algebraical Functions.*

To integrate any *rational function* M/D or $\mathrm{f}x/\mathrm{F}x$, the numerator and denominator of which are *rational integral algebraical functions* of x (§ 15), and therefore of the form

$$\frac{\mathrm{f}x}{\mathrm{F}x}=\frac{ax^m+bx^{m-1}+cx^{m-2}+\ldots}{Ax^n+Bx^{n-2}+Cx^{n-3}+\ldots}$$

where m and n are positive integers, the function is first resolved into its *partial fractions* by the ordinary rules of Algebra (Smith, *Algebra*, chap. xxiii.; Hall and Knight, *Higher Algebra*, chap. xxiii.); if the degree m of the numerator is equal to or greater than the degree n of the denominator, the *quotient* must be first obtained by division.

Thus to integrate $\dfrac{x^3}{x^2-3x+2}$, we must suppose it resolved into the form

$$x+3+\frac{A}{x-1}+\frac{B}{x-2}\equiv\frac{x^3}{x^2-3x+2},$$

$x+3$ being the quotient and $A/(x-1)$, $B/(x-2)$ the partial fractions of the remainder.

To determine the numerator A, multiply both sides of the identity by its denominator $x-1$; then

$$(x-1)(x+3)+A+B\,\frac{x-1}{x-2}\equiv\frac{x^3}{x-2};$$

and now put $x-1=0$, in the identity: then $A=-1$.

Similarly to determine B, multiply both sides by its denominator $x-2$, and then put this denominator $x-2=0$; this gives $B=8$.

$$\text{Now}\quad\int\frac{x^3dx}{x^2-3x+2}=\int\left(x+3-\frac{1}{x-1}+\frac{8}{x-2}\right)dx$$
$$=\tfrac{1}{2}x^2+3x-\log(x-1)+8\log(x-2),$$

and the integration is effected.

Every rational integral algebraical function of x, such as Fx, can be resolved into real linear factors (factors of the first degree in x), or real quadratic factors (factors of the second degree); thus, for example,

$x^2-1=(x-1)(x+1)$;

x^2+1 is not decomposable into real linear factors;

$x^3-1=(x-1)(x^2+x+1)$;

$x^3+1=(x+1)(x^2-x+1)$;

$x^4-1=(x-1)(x+1)(x^2+1)$;

$x^4+1=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$;

$x^6-1=(x-1)(x+1)(x^2-x+1)(x^2+x+1)$;

$x^6+1=(x^2+1)(x^2-\sqrt{3}x+1)(x^2+\sqrt{3}x+1)$;

$x^8-1=(x-1)(x+1)(x^2+1)(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$;

$$x^8+1=\{x^2-\sqrt{(2+\sqrt{2})}x+1\}\{x^2+\sqrt{(2+\sqrt{2})}x+1\}\times \{x^2-\sqrt{(2-\sqrt{2})}x+1\}\{x^2+\sqrt{(2-\sqrt{2})}x+1\};$$

and so on.

Corresponding to a quadratic factor in the denominator Fx we must assume a partial fraction with a numerator of the form $Hx+K$; thus, as resolved into partial fractions, we must put $\dfrac{1}{x^3-1}=\dfrac{A}{x-1}+\dfrac{Hx+K}{x^2+x+1}$.

The numerator A is then determined as before by multiplying both sides by $x-1$ and then putting $x-1=0$; thus $A=\frac{1}{3}$; and now, by transposition,

$$\frac{Hx+K}{x^2+x+1}=\frac{1}{x^3-1}-\frac{1}{3}\frac{1}{x-1}=\frac{-x-2}{3(x^2+x+1)}$$

(so that $H=-\frac{1}{3}$, $K=-\frac{2}{3}$).

Now
$$\int\frac{dx}{x^3-1}=\int\left(\frac{1}{3}\frac{1}{x-1}-\frac{1}{3}\frac{x+2}{x^2+x+1}\right)dx$$
$$=\frac{1}{3}\log(x-1)-\frac{1}{6}\int\frac{2x+1}{x^2+x+1}dx-\frac{1}{2}\int\frac{dx}{x^2+x+1}$$
$$=\frac{1}{3}\log(x-1)-\frac{1}{6}\log(x^2+x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$$
$$=\frac{1}{6}\log\frac{(x-1)^3}{x^3-1}-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

Examples.—Resolve into partial fractions and integrate

(1) $\frac{1,\ x,\ x^2,\ x^3}{(x-1)(x-2)}$, $\frac{1,\ x,\ x^2,\ x^3}{(x-a)(x-b)}$, $\frac{1,\ x,\ x^2, \dots}{(x-1)(x-2)(x-3)}$,

$\frac{1,\ x,\ x^2, \dots}{(x-a)(x-b)(x-c)}$, $\frac{1,\ x,\ x^2, \dots}{(x^2+a^2)(x^2+b^2)}$.

(2) $\frac{1,\ x,\ x^2,\ x^3, \dots}{x^2-1,\ x^2+1,\ x^3-1,\ x^3+1}$.

(3) $\frac{Ax+B}{(mx+n)^r}$ (substitute $mx+n=y$),

$\frac{x^p}{(x-a)^m(x-b)^n}$ (substitute $\frac{x-a}{x-b}=y$),

$\frac{1}{x(ax^m+c)^n}$ (substitute $ax^m+c=1/y$),

$\frac{1}{(x+1)(x^2-1)}$, $\frac{1,\ x,\ x^2}{(x+1)^2(x^2+1)}$.

(4) $\frac{x^2-a^2}{x^4+a^2x^2+a^4}$, $\frac{1}{x(1+13x^2+36x^4)}$.

188. Generally, if $x-p$ denotes a real linear factor of the denominator Fx, so that

$$\mathrm{F}x=(x-p)\phi x,$$

we put
$$\frac{\mathrm{f}x}{\mathrm{F}x}=\frac{A}{x-p}+\frac{R}{\phi x},$$

and now
$$A=\frac{\mathrm{f}x-R(x-p)}{\phi x}=\frac{\mathrm{f}p}{\phi p},$$

on putting $x-p=0$.

But
$$(x-p)\phi x=\mathrm{F}x,$$
$$\phi x+(x-p)\phi' x=\mathrm{F}'x;$$

so that
$$\phi p=\mathrm{F}'p;\ \text{and}\ A=\mathrm{f}p/\mathrm{F}'p.$$

Now the integral of the corresponding partial fraction $A/(x-p)$ is $A\log(x-p)$.

If $(x-a)^2+\beta^2$ denotes a quadratic factor of Fx, splitting up into the conjugate imaginary linear factors

$$x-a-i\beta \text{ and } x-a+i\beta,$$

we take two corresponding partial fractions

$$\frac{L+iM}{x-a-i\beta}+\frac{L-iM}{x-a+i\beta},$$

coalescing into the real partial fraction

$$\frac{2L(x-a)-2M\beta}{(x-a)^2+\beta^2},$$

the integral of which is

$$L\log\{(x-a)^2+\beta^2\}-2M\tan^{-1}\{(x-a)/\beta\}.$$

We generally begin by assuming

$$\frac{\mathrm{f}x}{\mathrm{F}x}=\frac{Hx+K}{(x-a)^2+\beta^2}+\frac{S}{\psi x},$$

where $\mathrm{F}x=\{(x-a)^2+\beta^2\}\psi x$;

and $Hx+K=\dfrac{\mathrm{f}x-S\{(x-a)^2+\beta^2\}}{\psi x}=\dfrac{\mathrm{f}x}{\psi x}$,

on putting $(x-a)^2+\beta^2=0$.

This gives imaginary values of x; but we may avoid the use of imaginaries by continually writing $2ax-a^2-\beta^2$ for x^2, when finally fx and $(Hx+K)\psi x$ are each reduced to the form $Bx+C$ and $B'x+C'$; and thus $B=B'$, $C=C'$, whence H and K are determined.

189. When Fx has a factor $(x-q)^r$, a linear factor $x-q$ repeated r times, so that $\mathrm{F}x=(x-q)^r\chi x$, we must assume r corresponding partial functions of the form

$$\frac{B_r}{(x-q)^r}+\frac{B_{r-1}}{(x-q)^{r-1}}+\dots+\frac{B_1}{x-q}=\frac{\mathrm{f}x}{\mathrm{F}x}-\frac{T}{\chi x};$$

and then to determine the B's put $x-q=y$, or $x=q+y$; so that multiplying by y^r,

$$B_r+B_{r-1}y+\dots+B_1y^{r-1}=\{\mathrm{f}(q+y)-Ty^r\}/\chi(q+y);$$

$B_r, B_{r-1}, \dots B_1$ are thus the coefficients of $y^0, y^1, \dots y^{r-1}$

in the algebraical expansion of $\mathrm{f}(q+y)/\chi(q+y)$ in ascending powers of y; or, in other words,

$$B_r+B_{r-1}y+\ldots+B_1y^{r-1}$$

is the quotient and Ty^r is the remainder, when $\mathrm{f}(q+y)$ is divided algebraically by $\chi(q+y)$ in ascending powers of y.

Now the integral of a partial fraction $B_s(x-q)^{-s}$ will be $\dfrac{-B_s}{(s-1)(x-q)^{s-1}}$; and of $\dfrac{B_1}{x-q}$ will be $B_1\log(x-q)$.

When $\mathrm{F}x$ has the repeated factor $\{(x-a)^2+\beta^2\}^r$, we can proceed in the same way, and assume r corresponding partial fractions of the form

$$(H_sx+K_s)/\{(x-a)^2+\beta^2\}^s;$$

but in this case it is generally preferable to employ the conjugate imaginary factors; or else to employ the substitution $\quad x-a=\beta\tan\theta$ or $\beta\sinh u$.

190. Consider $\displaystyle\int\frac{x^{m-1}dx}{x^n\pm1}$, where m and n are positive integers, required in the vertical motion of a body when the resistance varies as the nth power of the velocity.

If $x-a$ is a factor of x^n-1, then

$$a=1, \text{ or } \cos(2r\pi/n)+i\sin(2r\pi/n)=\exp(2ir\pi/n),$$

where r has the values ±1, ±2,...

For then, by De Moivre's Theorem (§ 111),

$$a^n=\cos 2r\pi+i\sin 2r\pi=1.$$

Put $\quad \dfrac{x^{m-1}}{x^n-1}=\dfrac{A}{x-1}+\Sigma\dfrac{A_r}{x-a}$;

here $\quad \mathrm{f}x=x^{m-1}$, $\mathrm{F}x=x^n-1$, $\mathrm{F}'x=nx^{n-1}$;

so that $\quad A_r=\dfrac{\mathrm{f}a}{\mathrm{F}'a}=\dfrac{a^{m-1}}{na^{n-1}}=\dfrac{a^m}{na^n}=\dfrac{a^m}{n}$ (since $a^n=1$)

$$=\frac{1}{n}\left(\cos\frac{2mr\pi}{n}+i\sin\frac{2mr\pi}{n}\right)=\frac{1}{n}\exp\frac{2imr\pi}{n},$$

by De Moivre's Theorem; also $A=1/n$.

Therefore $$\int \frac{x^{m-1}dx}{x^n-1} = \frac{1}{n}\log(x-1)$$

$$+\frac{1}{n}\Sigma\left(\cos\frac{2mr\pi}{n}+i\sin\frac{2mr\pi}{n}\right)\log\left(x-\cos\frac{2r\pi}{n}-i\sin\frac{2r\pi}{n}\right).$$

To express this result in a real form, the partial fractions with numerator A_r and A_{-r}, corresponding to conjugate imaginary values of a, must be combined into a real form as follows—

$$\frac{A_r}{x-\cos(2r\pi/n)-i\sin(2r\pi/n)}+\frac{A_{-r}}{x-\cos(2r\pi/n)+i\sin(2r\pi/n)}$$

$$=\frac{1}{n}\cos\frac{2mr\pi}{n}\;\frac{2x-2\cos(2r\pi/n)}{x^2-2x\cos(2r\pi/n)+1}$$

$$-\frac{2}{n}\sin\frac{2mr\pi}{n}\;\frac{\sin(2r\pi/n)}{x^2-2x\cos(2r\pi/n)+1};$$

and the corresponding integrals are

$$\frac{1}{n}\cos\frac{2mr\pi}{n}\log\left(x^2-2x\cos\frac{2r\pi}{n}+1\right)$$

$$-\frac{2}{n}\sin\frac{2mr\pi}{n}\tan^{-1}\left\{\left(x-\cos\frac{2r\pi}{n}\right)\Big/\sin\frac{2r\pi}{n}\right\}.$$

When the denominator is x^n+1, the result is of the same form, with $2r-1$ written for $2r$; and from the above we infer that the typical quadratic factors of x^n-1 and x^n+1 are

$$x^2-2x\cos(2r\pi/n)+1, \text{ and } x^2-2x\cos\{(2r-1)\pi/n\}+1.$$

If m is changed into $n-m$, as for instance by the substitution $x=1/y$, then $\cos(2mr\pi/n)$ is unchanged, but $\sin(2mr\pi/n)$ changes in sign, so that (Euler)

$$\int \frac{x^{m-1}\pm x^{n-m-1}}{x^n\pm 1}dx$$

is expressed entirely, either by logarithms, or inverse circular functions.

Degenerate cases occur for $n=m$, or $2m$.

When n is even, $=2p$, and m is odd, $=2q+1$, then, by a rearrangement of terms,

$$\int \frac{x^{m-1}dx}{x^n+1}=\int \frac{x^{2q}dx}{x^{2p}+1}$$

$$=\frac{1}{2p}\sum_{r=1}^{r=n}\cos\frac{(2r-1)(2q+1)\pi}{2p}\tanh^{-1}\frac{2x\cos(2r-1)\pi/2p}{x^2+1}$$

$$-\frac{1}{2p}\sum\sin\frac{(2r-1)(2q+1)\pi}{2p}\tan^{-1}\frac{2x\sin(2r-1)\pi/2p}{x^2-1}.$$

Taken between the limits 0 and ∞, the part depending on the hyperbolic functions vanishes; and the result is

$$\int_0^\infty \frac{x^{2q}dx}{x^{2p}+1}=\frac{\pi}{2p}\sum_{r=1}^{r=p}\sin\frac{(2r-1)q\pi}{p}=\frac{\pi}{2p}\operatorname{cosec}\frac{2q+1}{2p}\pi.$$

We may put $x^{2p}=v^n$, and $(2q+1)/2p=m/n$, where m and n are any integers; and now, with the restriction that $m<n$,

$$\int_0^\infty \frac{v^{m-1}dv}{v^n+1}=\frac{\pi}{n}\operatorname{cosec}\frac{m\pi}{n}.$$

Thus, for instance, the area of the loop of the curve

$$x^n+y^n=a^2x^{n-m-1}y^{m-1}$$

on putting $y=xv$ (§ 63), is given by

$$\tfrac{1}{2}\int_0^\infty x^2dv=\tfrac{1}{2}a^2\int_0^\infty \frac{v^{m-1}dv}{1+v^n}=\frac{\pi a^2}{2n}\operatorname{cosec}\frac{m\pi}{n}.$$

Putting $v^n=t$, then with the restriction that $m<1$,

$$\int_0^\infty \frac{t^{m-1}dy}{1+t}=\pi\operatorname{cosec} m\pi.$$

To determine $\int_0^{\frac{1}{2}\pi}(\tan\theta)^{q/p}d\theta$, put $\tan\theta=x^p$; then the integral becomes

$$\int_0^\infty \frac{px^{p+q-1}dx}{1+x^{2p}}=\tfrac{1}{2}\pi\operatorname{cosec}\frac{p+q}{2p}\pi.$$

Similarly for $\int_0^\infty (\tanh u)^{q/p}du.$

Examples.—Resolve into partial fractions and integrate

(1) $\dfrac{1,\ x,\ x^2,\ x^3\ldots}{x^n-1 \text{ or } x^n+1}$, for $n=4,\ 6,\ 8$.

(2) $\dfrac{(1+x^2)^2}{(1-x^2)^3},\ \dfrac{(1+x^2)^4}{(1-x^2)^5},\ldots,\ \dfrac{(1-x^2)^2}{(1+x^2)^3},\ \dfrac{(1-x^2)^4}{(1+x^2)^5},\ldots$

(the integrations required when we integrate $(\sec\theta)^3$, $(\sec\theta)^5$, ... or $(\cosh u)^2$, $(\cosh u)^4$, ... or $\cos^2\theta$, $\cos^4\theta$; ... or $(\operatorname{sech} u)^3$, $(\operatorname{sech} u)^5$, ... by means of the substitution $\tan\frac{1}{2}\theta$ or $\tanh\frac{1}{2}u=x$.

191. *Integration of an Irrational Algebraical Function.* In integrating the irrational part $N/D\sqrt{R}$ we may suppose the rational function N/D resolved into its quotient and partial fractions; and now we shall consider the integration of the simplest elements,

$$\frac{1}{\sqrt{R}},\quad \frac{1}{(x-p)\sqrt{R}},\quad \text{and}\quad \frac{Hx+K}{(Ax^2+2Bx+C)\sqrt{R}},$$

corresponding to the constant term in the quotient of N/D, and to the partial fractions corresponding with the linear and quadratic factors $x-p$ and $Ax^2+2Bx+C$ of the denominator D; and afterwards investigate *formulas of reduction* for

$$\frac{x^n}{\sqrt{R}},\quad \frac{1}{(x-q)^r\sqrt{R}},\quad \text{and}\quad \frac{Hx+K}{(Ax^2+2Bx+C)^s\sqrt{R}}.$$

By formulas (i), (w), (x) (§ 39),

$$\text{(i.)}\quad \int_0^x \frac{dx}{\sqrt{(m^2-x^2)}} = \sin^{-1}\frac{x}{m} = \cos^{-1}\sqrt{\left(1-\frac{x^2}{m^2}\right)};$$

$$\text{(ii.)}\quad \int_0^x \frac{dx}{\sqrt{(m^2+x^2)}} = \sinh^{-1}\frac{x}{m} = \cosh^{-1}\sqrt{\left(1+\frac{x^2}{m^2}\right)};$$

$$\text{(iii.)}\quad \int_m^x \frac{dx}{\sqrt{(x^2-m^2)}} = \cosh^{-1}\frac{x}{m} = \sinh^{-1}\sqrt{\left(\frac{x^2}{m^2}-1\right)};$$

in which it will be noticed that the analogy with the first integral breaks down when only the logarithm

$$\log\left\{\sqrt{\left(1+\frac{x^2}{m^2}\right)}+\frac{x}{m}\right\} \text{ or } \log\left\{\frac{x}{m}+\sqrt{\left(\frac{x^2}{m^2}-1\right)}\right\}$$

is employed to express the second and third integral.

The quadratic $R=ax^2+2bx+c$ can always be expressed as the sum or difference of two squares in one of the three forms m^2-x^2, m^2+x^2, or x^2-m^2; and then the integral $\int dx/\sqrt{R}$ has the corresponding form (i.), (ii.), or (iii.).

But substituting $R=y^2$, we find

$$ax+b=\sqrt{(ay^2+b^2-ac)};$$

so that
$$\int\frac{dx}{\sqrt{R}}=\int\frac{dy}{\sqrt{(ay^2+b^2-ac)}};$$

and now, as in ex. 13, p. 77,

(i.) when a is negative, but b^2-ac positive,

$$\int\frac{dx}{\sqrt{R}}=\frac{1}{\sqrt{(-a)}}\sin^{-1}\frac{\sqrt{(-a)}y}{\sqrt{(b^2-ac)}}$$
$$=\frac{1}{\sqrt{(-a)}}\sin^{-1}\frac{\sqrt{(-a)}\sqrt{R}}{\sqrt{(b^2-ac)}}=\frac{1}{\sqrt{(-a)}}\cos^{-1}\frac{ax+b}{\sqrt{(b^2-ac)}};$$

(ii.) when a is positive, and b^2-ac positive,

$$\int\frac{dx}{\sqrt{R}}=\frac{1}{\sqrt{(a)}}\sinh^{-1}\frac{\sqrt{(a)}\sqrt{R}}{\sqrt{(b^2-ac)}}=\frac{1}{\sqrt{(a)}}\cosh^{-1}\frac{ax+b}{\sqrt{(b^2-ac)}};$$

(iii.) when a is positive, and b^2-ac negative,

$$\int\frac{dx}{\sqrt{R}}=\frac{1}{\sqrt{(a)}}\cosh^{-1}\frac{\sqrt{(a)}\sqrt{R}}{\sqrt{(ac-b^2)}}=\frac{1}{\sqrt{(a)}}\sinh^{-1}\frac{ax+b}{\sqrt{(ac-b^2)}}.$$

We cannot have both a and b^2-ac negative, because R would then be negative, and $\sqrt{R}$ imaginary, for all real values of x.

When $a=0$ the integral assumes an indeterminate form (§ 120), and reduces to an algebraical function

$$\sqrt{R}/b, \text{ or } \sqrt{(2bx+c)}/b.$$

When b^2-ac is positive, R breaks up into real linear factors, say $x-\alpha$ and $x-\beta$; and now

(1) when a is negative, we may without loss of generality put $a=-1$; and

$$\int\frac{dx}{\surd R}=\int\frac{dx}{\surd\{(\alpha-x)(x-\beta)\}}=\sin^{-1}\frac{\surd(\alpha-x\,.\,x-\beta)}{\frac{1}{2}(\alpha-\beta)}$$

$$=2\sin^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}}=2\cos^{-1}\sqrt{\frac{\alpha-x}{\alpha-\beta}}=2\tan^{-1}\sqrt{\frac{x-\beta}{\alpha-x}}.$$

(2) when a is positive, we may put $a=1$, and

$$\int\frac{dx}{\surd R}=\int\frac{dx}{\surd(x-\alpha\,.\,x-\beta)}=\sinh^{-1}\frac{\surd(x-\alpha\,.\,x-\beta)}{\frac{1}{2}(\alpha-\beta)}$$

$$=2\sinh^{-1}\sqrt{\frac{x-\alpha}{\alpha-\beta}}=2\cosh^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}}=2\tanh^{-1}\sqrt{\frac{x-\alpha}{x-\beta}}$$

$$=2\log\frac{\surd(x-\alpha)+\surd(x-\beta)}{\surd(\alpha-\beta)}.$$

We may write, by analogy,

$$\int\frac{dx}{R}=\frac{1}{\surd(ac-b^2)}\quad\sec^{-1}\frac{\surd a\surd R}{\surd(ac-b^2)},$$

or $$-\frac{1}{\surd(b^2-ac)}\quad\text{sech}^{-1}\frac{\surd(-a)\surd R}{\surd(b^2-ac)},$$

or $$-\frac{1}{\surd(b^2-ac)}\text{cosech}^{-1}\frac{\surd a\surd R}{\surd(b^2-ac)},$$

thus exhibiting the result as a function of R.

For instance

$$\int_x^\infty\frac{dx}{x^2-2ax\cos\alpha+a^2}=\frac{1}{a\sin\alpha}\,\text{cosec}^{-1}\frac{\surd(x^2-2ax\cos\alpha+a^2)}{a\sin\alpha};$$

$$\int_x^\infty\frac{dx}{x^2-2ax\cosh\beta+a^2}=\frac{1}{a\sinh\beta}\text{cosech}^{-1}\frac{\surd(x^2-2ax\cosh\beta+a^2)}{a\sinh\beta};$$

$$\int_x^\infty\frac{dx}{x^2-2ax\sinh\gamma-a^2}=\frac{1}{a\cosh\gamma}\text{cosech}^{-1}\frac{\surd(x^2-2ax\sinh\gamma-a^2)}{a\cosh\gamma}.$$

192. Next to determine $\int \frac{dx}{(x-p)\sqrt{R}}$, we may substitute $x-p=1/y$, when we obtain an integral of the same form as $\int dx/\sqrt{R}$; but it is more direct to substitute $y=\sqrt{R}/(x-p)$; and now

$$y^2+\lambda=\frac{R+\lambda(x-p)^2}{(x-p)^2}=\frac{(a+\lambda)x^2+2(b-p\lambda)x+c+\lambda p^2}{(x-p)^2};$$

a perfect square when $\lambda=\frac{b^2-ac}{ap^2+2bp+c}$; so that

$$(ap^2+2bp+c)y^2+b^2-ac=\left\{\frac{(ap+b)x+bp+c}{x-p}\right\}^2;$$

and taking the negative sign with the radical

$$\frac{(ap+b)x+bp+c}{x-p}=-\sqrt{\{(ap^2+2bp+c)y^2+b^2-ac\}}$$

$$\frac{dx}{(x-p)^2}=\frac{ydy}{\sqrt{\{(ap^2+2bp+c)y^2+b^2+ac\}}}$$

$$\int\frac{dx}{(x-p)\sqrt{R}}=\int\frac{dx}{(x-p)^2y}=\int\frac{dy}{\sqrt{\{(ap^2+2bp+c)y^2+b^2-ac\}}}$$

$$=\frac{1}{\sqrt{(-ap^2-2bp-c)}}\sin^{-1}\frac{\sqrt{(-ap^2-2bp-c)}y}{\sqrt{(b^2-ac)}}$$

$$=\frac{1}{\sqrt{(-ap^2-2bp-c)}}\sin^{-1}\frac{\sqrt{(-ap^2-2bp-c)}\sqrt{R}}{\sqrt{(b^2-ac)}(x-p)}\text{....(i.)};$$

$$\text{or }=\frac{1}{\sqrt{(ap^2+2bp+c)}}\sinh^{-1}\frac{\sqrt{(ap^2+2bp+c)}\sqrt{R}}{\sqrt{(b^2-ac)}(x-p)}\text{.....(ii.)};$$

$$\text{or }=\frac{1}{\sqrt{(ap^2+2bp+c)}}\cosh^{-1}\frac{\sqrt{(ap^2+2bp+c)}\sqrt{R}}{\sqrt{(ac-b^2)}(x-p)}\text{....(iii.)};$$

the real form to be chosen (ex. 18, p. 78).

Here again, when $ap^2+2bp+c=0$, so that $x-p$ is a factor of R, the integral assumes an indeterminate form and the limiting value is the algebraical function

$$\frac{\sqrt{R}}{\sqrt{(b^2-ac)(x-p)}}.$$

193. We may consider the integral $\int dx/(x-p)\sqrt{R}$ as the degenerate form of the *elliptic integral* $\int dx/\sqrt{X}$, where X is a quartic function of x, as in ex. 39, p. 82, when X splits up into the factors $(x-p)^2$ and R.

Similarly the *canonical* elliptic integral $\int ds/\sqrt{S}$, where, as in ex. 39, p. 82,

$$S=4s^3-g_2s-g_3=4(s-e_1)(s-e_2)(s-e_3),$$

when resolved into factors, becomes one of the degenerate forms of exs. 14, 15, p. 77, if two of the quantities e_1, e_2, e_3 are equal.

We suppose $e_1>e_2>e_3$; and that the middle quantity e_2 is made equal either to the smallest e_3, or the greatest e_1; and then with (i.) $e_2=e_3$, as in ex. 14, p. 77,

$$\int\frac{ds}{\sqrt{S}}=\int\frac{ds}{2(x-e_3)\sqrt{(x-e_1)}}$$

$$=\frac{1}{\sqrt{(e_1-e_3)}}\sin^{-1}\sqrt{\frac{x-e_1}{x-e_3}}=\frac{1}{2\sqrt{(e_1-e_3)}}\sin^{-1}\frac{2\sqrt{(e_1-e_3.x-e_1)}}{x-e_3};$$

(ii) $e_2=e_1$, $\int\frac{ds}{\sqrt{S}}$

$$=\int\frac{ds}{2(e_1-x)\sqrt{(x-e_3)}}=\frac{1}{\sqrt{(e_1-e_3)}}\sinh^{-1}\sqrt{\frac{x-e_3}{e_1-x}}\ (x<e_1),$$

or

$$=\frac{1}{\sqrt{(e_1-e_3)}}\cosh^{-1}\sqrt{\frac{x-e_3}{x-e_1}}\ (x>e_1).$$

Similarly, as in ex. 15, p. 77,

$$\int\frac{ds}{2(e_1-x)\sqrt{(e_3-x)}}=\frac{1}{\sqrt{(e_1-e_3)}}\cos^{-1}\sqrt{\frac{e_3-x}{e_1-x}};$$

$$\int\frac{ds}{2(e_3-x)\sqrt{(e_1-x)}}=\frac{1}{\sqrt{(e_1-e_3)}}\cosh^{-1}\sqrt{\frac{e_1-x}{e_3-x}}\ (x<e_3),$$

or

$$=\frac{1}{\sqrt{(e_1-e_3)}}\sinh^{-1}\sqrt{\frac{e_1-x}{x-e_3}}\ (x>e_3).$$

In physical problems we require the integrals

$$A=\int_\lambda^\infty \frac{d\lambda}{(a^2+\lambda)^2(c^2+\lambda)^{\frac{1}{2}}} \text{ and } C=\int_\lambda^\infty \frac{d\lambda}{(a^2+\lambda)(c^2+\lambda)^{\frac{3}{2}}}.$$

Now $$\int_\lambda^\infty \frac{d\lambda}{(a^2+\lambda)\sqrt{(c^2+\lambda)}}=\frac{2}{\sqrt{(a^2-c^2)}}\cos^{-1}\sqrt{\frac{c^2+\lambda}{a^2+\lambda}},$$

or $$=\frac{2}{\sqrt{(c^2-a^2)}}\cosh^{-1}\sqrt{\frac{c^2+\lambda}{a^2+\lambda}};$$

and, differentiating with respect to a^2 or c^2, we deduce

$$A=-\frac{\sqrt{(c^2+\lambda)}}{(a^2-c^2)(a^2+\lambda)}+\frac{1}{(a^2-c^2)^{\frac{3}{2}}}\cos^{-1}\sqrt{\frac{c^2+\lambda}{a^2+\lambda}},$$

or $$-\frac{\sqrt{(c^2+\lambda)}}{(c^2-a)(a^2+\lambda^2)}+\frac{1}{(c^2-a^2)^{\frac{3}{2}}}\cosh^{-1}\sqrt{\frac{c^2+\lambda}{a^2+\lambda}};$$

$$C=\frac{2}{(a^2-c^2)\sqrt{(c^2+\lambda)}}-\frac{2}{(a^2-c^2)^{\frac{3}{2}}}\cos^{-1}\sqrt{\frac{c^2+\lambda}{a^2+\lambda}},$$

or $$\frac{2}{(c^2-a^2)\sqrt{(c^2+\lambda)}}-\frac{2}{(c^2-a^2)^{\frac{3}{2}}}\cosh^{-1}\sqrt{\frac{c^2+\lambda}{a^2+\lambda}};$$

so that $$2A+C=\frac{2}{(a^2+\lambda)\sqrt{(c^2+\lambda)}}.$$

194. To determine the integral

$$\int\frac{Hx+K}{Ax^2+2Bx+C}\frac{dx}{\sqrt{(ax^2+2bx+c)}},$$

where we may suppose A and therefore $Ax^2+2Bx+C$ positive for all real values of x, substitute

$$y=(ax^2+2bx+c)/(Ax^2+2Bx+C).$$

Now, with the notation of ex. 8, p. 146,

$$\frac{dy}{dx}=\frac{2(Ab-aB)(x_1-x)(x-x_2)}{(Ax^2+2Bx+C)^2},$$

$$y_1-y=\frac{(Ay_1-a)(x_1-x)^2}{Ax^2+2Bx+C};\quad y-y_2=\frac{(a-Ay_2)(x-x_2)^2}{Ax^2+2Bx+C};$$

where y_1, y_2 denote the maximum and minimum of y, and x_1, x_2 the corresponding values of x.

We may write $L(x-x_2)+M(x_1-x)$ for $Hx+K$, and

$$V=Ax^2+2Bx+C=P(x_1-x)^2+Q(x-x_2)^2,$$
$$R=ax^2+2bx+c=p(x_1-x)^2+q(x-x_2)^2,$$

as before, on p. 147; and now we find

$$\int\frac{L(x-x_2)+M(x_1-x)}{V\surd R}dx$$
$$=\int\frac{L(x-x_2)+M(x_1-x)}{V^{\frac{3}{2}}\surd y}\;\frac{V^2dy}{2(Ab-aB)(x_1-x)(x-x_2)}$$
$$=\tfrac{1}{2}LL'\int\frac{dy}{\surd\{y(y_1-y)\}}+\tfrac{1}{2}MM'\int\frac{dy}{\surd\{y(y-y_2)\}}$$

where $L'=\dfrac{\surd(Ay_1-a)}{Ab-aB}$, $M'=\dfrac{\surd(a-Ay_2)}{Ab-aB}$;

so that the expression for the integral is

$$-LL'\cos^{-1}\surd(y/y_1)+MM'\cosh^{-1}\surd(y/y_2),$$

or $$LL'\sin^{-1}\surd(y/y_1)+MM'\sinh^{-1}\surd(-y/y_2),$$

according as y_2 is positive or negative; that is, according as $ac-b^2$ is positive or negative; that is, as R is always positive, or as R can vanish, for real values of x.

When $V=R$, these forms become illusory; and now

$$\int R^{-\frac{3}{2}}dx=(ax+b)/(ac-b^2)\surd R,$$
$$\int(ax+b)R^{-\frac{3}{2}}dx=R^{-\frac{1}{2}};$$

whence $\int(Hx+K)R^{-\frac{3}{2}}dx$ can be determined, as an algebraical function of x; and similarly $\int(Hx+K)R^{-\frac{5}{2}}dx,\dots.$

Again, when $Ab-aB=0$, the results are again illusory; but in this case we can choose a new variable, by changing $x+b/a$ or $x+B/A$ into x, so that we may make b and B vanish; and now the integral

$$\int\frac{Lx+M}{Ax^2+C}\frac{dx}{\surd(ax^2+c)}$$

consists of two parts, of which the first is a function of x^2, of the form given in § 193; while, as in ex. 19, p. 79,

$$\int \frac{M\,dx}{(Ax^2+C)\sqrt{(ax^2+c)}} = \frac{M}{\sqrt{C}\sqrt{(Ac-aC)}}\cos^{-1}\sqrt{\left(\frac{C}{c}\,\frac{ax^2+c}{Ax^2+C}\right)},$$

or
$$\frac{M}{\sqrt{C}\sqrt{(aC-Ac)}}\cosh^{-1}\sqrt{\left(\frac{C}{c}\,\frac{ax^2+c}{Ax^2+C}\right)},$$

or
$$\frac{M}{\sqrt{C}\sqrt{(aC-Ac)}}\sinh^{-1}\sqrt{\left(-\frac{C}{c}\,\frac{ax^2+c}{Ax^2+C}\right)};$$

reducing, when $aC-Ac=0$, to $\dfrac{x}{C\sqrt{(ax^2+c)}}$.

By differentiation of the integral

$$\int \frac{Hx+K}{Ax^2+2Bx+C}\,\frac{dx}{\sqrt{(ax^2+2bx+c)}}$$

with respect to A, B, or C, we can deduce the results of

$$\int \frac{\mathrm{f}x}{(Ax^2+2Bx+C)^n}\,\frac{dx}{\sqrt{(ax^2+2bx+c)}}.$$

The general linear substitution may be written

$$x' = \frac{e(x_1-x)+f(x-x_2)}{E(x_1-x)+F(x-x_2)},$$

and the form of the integral will then be unchanged; in particular V and R are interchanged if we put

$$E=\sqrt{(Pp)},\ F=\sqrt{(Qq)},\ e=Ex_1,\ f=Fx_2.$$

Examples.

(1) Integrate

$$\frac{x-1}{(x^2+x+1)\sqrt{(x^2-x+1)}},\quad \frac{x+1}{(x^2-x+1)\sqrt{(x^2+x+1)}},$$

$$\frac{L(x-1)+M(x-2)}{(3x^2-10x+9)\sqrt{(5x^2-16x+14)}},$$

$$\frac{L(x-1)+M(x-2)}{(5x^2-16x+14)\sqrt{(3x^2-10x+9)}}$$

$$\frac{Hx+K}{(3x^2-10x+9)\sqrt{(x^2-8x+10)}}.$$

(2) Prove that the maximum and minimum of

$$(ax^2+2hxy+by^2)/(Ax^2+2Hxy+By^2)$$

are given by the common conjugate diameters of

$$ax^2+2hxy+by^2=c,\ Ax^2+2Hxy+By^2=C.$$

(3) With the notation of ex. 8, p. 146. prove that

$$\begin{aligned} ax+b &= -p(x_1-x)+\quad q(x-x_2), \\ bx+c &= \ px_1(x_1-x)-qx_2(x-x_2); \\ Ax+B &= -P(x_1-x)+\quad Q(x-x_2), \\ Bx+C &= \ Px_1(x_1-x)-Qx_2(x-x_2); \end{aligned}$$

and $(B^2-AC)R^2+(Ac+aC-2Bb)RV+(b^2-ac)V^2$

$$=(Ab-aB)^2(x_1-x)^2(x-x_2)^2.$$

If $x'=-(Bx+C)/(Ax+B)$, or $-(bx+c)/(ax+b)$,

$$\frac{R}{V}+\frac{R'}{V'}=\frac{Ac+aC-2Bb}{AC-B^2}, \text{ or } \frac{V}{R}+\frac{V'}{R'}=\frac{Ac+aC-2Bb}{ac-b^2}.$$

(4) Prove that if X denotes the *reciprocal* quartic

$$ax^4+4bx^3+6cx^2+4bx+a,$$

$$\text{(i.)} \int\frac{x^2-1}{x}\frac{dx}{\sqrt{X}}=\frac{1}{\sqrt{(-a)}}\sin^{-1}\frac{\sqrt{(-a)}\sqrt{X}}{\sqrt{(-D)}x},$$

$$\text{or } \frac{1}{\sqrt{a}}\sinh^{-1}\frac{\sqrt{a}\sqrt{X}}{\sqrt{(-D)}x}, \text{ or } \frac{1}{\sqrt{a}}\cosh^{-1}\frac{\sqrt{a}\sqrt{X}}{\sqrt{(D)}x};$$

where $aX=(ax^2+2bx+a)^2+Dx^2$, $D=6ac-4b^2-2a^2$.

$$\text{(ii.)} \int\frac{x^2-1}{x^2+1}\frac{dx}{\sqrt{X}}=\frac{1}{\sqrt{(2a-6c)}}\sin^{-1}\frac{\sqrt{(2a-6c)}\sqrt{X}}{\sqrt{(-D)}(x^2+1)},$$

$$\text{or } \frac{1}{\sqrt{(6c-2a)}}\sinh^{-1}\frac{\sqrt{(6c-2a)}\sqrt{X}}{\sqrt{(-D)}(x^2+1)},$$

$$\text{or } \frac{1}{\sqrt{(6c-2a)}}\cosh^{-1}\frac{\sqrt{(6c-2a)}\sqrt{X}}{\sqrt{(D)}(x^2+1)};$$

and $(6c-2a)X=4\{bx^2-(a-3c)x+b\}^2+D(x^2+1)^2$.

(5) Determine in a similar form

$$\int\frac{x^2+1}{x}\frac{dx}{\sqrt{X'}}, \text{ and } \int\frac{x^2+1}{x^2-1}\frac{dx}{\sqrt{X'}},$$

where $X'=ax^4-4bx^3+6cx^2+4bx+a$.

195. *Integration by Rationalization.*

Any integral of the form

$$\int \frac{S+T(ax+b)^{p/r}}{U+V(ax+b)^{q/r}}dx,$$

where S, T, U, V are rational integral functions of x, can be *rationalized*, that is, can be made to depend on the integration of a rational algebraical fraction by the substitution $ax+b=y^r$; for then $adx=ry^{r-1}dy$; and the integral becomes the rational algebraical integral

$$\int \frac{S+Ty^p}{U+Vy^q}\frac{ry^{r-1}}{a}dy.$$

Any integral of the form

$$\int \frac{S+T\sqrt{R}}{U+V\sqrt{R}}dx,$$

where R is the quadratic form $ax^2+2bx+c$, can be rationalized,

(i.) when b^2-ac is positive, and a negative, by the substitution

$$\frac{\sqrt{(-a)}\sqrt{R}}{\sqrt{(b^2-ac)}}=\frac{2y}{1+y^2}, \text{ then } \frac{ax+b}{\sqrt{(b^2-ac)}}=\frac{1-y^2}{1+y^2}$$

and $$\frac{-adx}{\sqrt{(b^2-ac)}}=\frac{4ydy}{(1+y^2)^2};$$

(ii.) when b^2-ac is positive, and a positive, by

$$\frac{\sqrt{(a)}\sqrt{R}}{\sqrt{(b^2-ac)}}=\frac{2y}{1-y^2}, \quad \frac{ax+b}{\sqrt{(b^2-ac)}}=\frac{1+y^2}{1-y^2},$$

and $$\frac{adx}{\sqrt{(b^2-ac)}}=\frac{4ydy}{(1-y^2)^2};$$

(iii.) when b^2-ac is negative, and a positive, by

$$\frac{\sqrt{(a)}\sqrt{R}}{\sqrt{(ac-b^2)}}=\frac{1+y^2}{1-y^2}, \quad \frac{ax+b}{\sqrt{(ac-b^2)}}=\frac{2y}{1-y^2},$$

and $$\frac{adx}{\sqrt{(ac-b^2)}}=2\frac{1+y^2}{(1-y^2)^2}dy.$$

In (i.) we may suppose $y=\tan\frac{1}{2}\theta$, and in (ii.) and (iii.) $y=\tanh\frac{1}{2}u$; and then the integration is changed to an integration of a rational function of $\cos\theta$ and $\sin\theta$, or $\cosh u$ and $\sinh u$, with respect to θ or u.

Examples.—Integrate with respect to x

$$\frac{x^3}{\surd(x-a)},\ \frac{x^n}{\surd(x+a)},\ x^n\surd(x-a),\ \frac{1}{(1+x)\surd x},$$

$$\frac{1}{x^4\surd(1+x^2)},\ \frac{1}{(x+1)\surd(x+2)},\ \frac{x}{(x+2)\surd(x+1)},$$

$$\frac{1}{(x+1)\surd(2x+1)},\frac{1}{(1-x)\surd(1-x^2)},\frac{1}{(1+x^2)\surd(1-x^2)},$$

$$\frac{\surd(a^2+x^2)}{a+x},\ \frac{1}{x\surd(x^2+3x+2)},\ \frac{1}{(x+1)\surd(x^2+x+1)},$$

$$\frac{1}{(x^2+1)^{\frac{3}{2}}},\ \frac{1}{(ax^2+2bx+c)^{\frac{3}{2}}},\ \frac{bx+c}{(ax^2+2bx+c)^{\frac{3}{2}}},$$

$$\frac{Hx+K}{(ax^2+2bx+c)^{\frac{3}{2}}},\ \frac{1}{x^3\surd(ax^2+c)},\ (1-2x^2)^{\frac{3}{2}},$$

$$\frac{x}{\surd(x^2+a^2.\,x^2+b^2)},\ \frac{1}{(x^4-1)^{\frac{1}{4}}},\ \surd(1+e^x),$$

$$(ax^n+b)^{-\frac{n+1}{n}},\ \text{(substitute } a+bx^{-n}=y\text{)}.$$

196. *Integration of Circular and Hyperbolic Functions.*

To integrate powers and products of $\cos x$ and $\sin x$ the most general plan is to convert them into cosines and sines of multiples of x, which are immediately integrable (ex. 9, § 40).

To integrate any odd power of $\cos x$ or $\sin x$, say $(\cos x)^{2n+1}$ or $(\sin x)^{2n+1}$, we write them in the form

$$(1-\sin^2 x)^n\cos x,\ \text{or}\ (1-\cos^2 x)^n\sin x,$$

and expand by the Binomial Theorem; then each term is immediately integrable, since by § 40 and (a), p. 84,

$$\int (\sin x)^m \cos x dx = \frac{(\sin x)^{m+1}}{m+1},$$

$$\int (\cos x)^m \sin x dx = -\frac{(\cos x)^{m+1}}{m+1}.$$

A similar method will serve to integrate $(\sin x)^p(\cos x)^q$, where either p or q is an odd integer; also to integrate any powers of vers x.

The same processes apply when $\cos x$ and $\sin x$ are replaced by the hyperbolic functions $\cosh x$ and $\sinh x$.

Examples.—Integrate with respect to x, $\cos mx$, $\sin(mx+n)$, $\sin 2x \cos 3x$, $\cos 3x \cos 5x$, $\sin 3x \sin 5x$, $\sin(mx+n)\cos(px+q)$, $\sin x \sin 2x \sin 3x$, $\sin x \cos x$, $\sin^2 x \cos x$, $(\sin x)^3 \cos x$, $(\sin x)^m \cos x$, $\sin x \cos^2 x$, $\sin x \cos^3 x$, $\sin x (\cos x)^m$, $\sin^2 x$, $\sin^3 x$, $\sin^4 x$, $\cos^2 x$, $\cos^3 x$, $\cos^4 x$, $\cos^2 mx \cos nx$, $\cos^3 mx \cos nx$, …, also the same functions with $\cosh x$ for $\cos x$, and $\sinh x$ for $\sin x$.

197. The integration of the remaining trigonometrical functions is a little more complicated; thus, by (v), p. 85,

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \log \sin x = \tfrac{1}{2} \log \text{vers } 2x;$$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\log \cos x = \log \sec x;$$

while $\int \coth x dx = \log \sinh x$, $\int \tanh x dx = \log \cosh x$.

Again $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x$,

$$\int \cot^2 x dx = \int (\operatorname{cosec}^2 x - 1) dx = -\cot x - x;$$

with similar results for $\tanh^2 x$ and $\coth^2 x$; while other powers of $\tan x$ are integrated by expressing them as the sum of terms of the form $(\sec x)^p \tan x$ or $(\tan x)^q \sec^2 x$, the integrals of which are $(\sec x)^p/p$ and $(\tan x)^{q+1}/(q+1)$; similarly for powers of $\cot x$, $\tanh x$, or $\coth x$.

To integrate $\sec x$ and $\operatorname{cosec} x$, we may rationalize by means of the substitution $\tan\frac{1}{2}x = y$; and then

$$\cos x = \frac{1-y^2}{1+y^2},\quad \sin x = \frac{2y}{1+y^2},$$

while

$$x = 2\tan^{-1}y,\quad dx = 2dy/(1+y^2).$$

Then

$$\int \sec x\,dx = \int \frac{1+y^2}{1-y^2}\,\frac{2dy}{1+y^2} = \int \frac{2dy}{1-y^2}$$

$$= \int\left(\frac{1}{1+y} + \frac{1}{1-y}\right)dy$$

$$= \log(1+y) - \log(1-y) = \log(1+y)/(1-y)$$

$$= \log\frac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x} = \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x) = \log\sqrt{\frac{1+\sin x}{1-\sin x}}$$

$$= \log(\sec x + \tan x).$$

Similarly

$$\int \operatorname{cosec} x\,dx = \int \frac{1+y^2}{2y}\,\frac{2dy}{1+y^2}$$

$$= \int dy/y = \log y = \log\tan\tfrac{1}{2}x = \log\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$= \log(\operatorname{cosec} x - \cot x).$$

Similarly to integrate $\operatorname{sech} x$ or $\operatorname{cosech} x$, we may rationalize by the substitution $\tanh\frac{1}{2}x = y$; and then

$$\int \operatorname{sech} x\,dx = \int \frac{2dy}{1+y^2} = 2\tan^{-1}y = 2\tan^{-1}\tanh\tfrac{1}{2}x,$$

$$\int \operatorname{cosech} x\,dx = \int dy/y \quad = \log y \quad = \quad \log\tanh\tfrac{1}{2}x.$$

The substitution of y for $\tan\frac{1}{2}\theta$ will apply for any function of the form $\mathrm{f}(\cos\theta, \sin\theta)/\mathrm{F}(\cos\theta, \sin\theta)$; the function being thereby reduced to the form

$$\frac{\Pi(y-b)}{\Pi(y-a)} \text{ or } \frac{\Pi\sin\frac{1}{2}(\theta-\beta)}{\Pi\sin\frac{1}{2}(\theta-a)};$$

and this again by partial fractions (§ 187) to the sum of terms of the form $A/(y-a)$, or

$$A\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}a\operatorname{cosec}\tfrac{1}{2}(\theta-a) = A\cos^2\tfrac{1}{2}a\cot\tfrac{1}{2}(\theta-a) - \tfrac{1}{2}A\sin a.$$

(Hermite, *Proc. London Math. Society*, vol. IV.).

198. By the substitution of z for $\sec x$, $\operatorname{cosec} x$, $\operatorname{sech} x$, $\operatorname{cosech} x$, the results can be expressed more directly; thus

with $$z = \sec x,\ dx = \frac{dz}{z\surd(z^2-1)},$$

and $$\int \sec x\,dx = \int \frac{dz}{\surd(z^2-1)} = \cosh^{-1} z, \text{ by (x.) (p. 85)}$$
$$= \cosh^{-1}\sec x = \sinh^{-1}\tan x = \tanh^{-1}\sin x = 2\tanh^{-1}\tan \tfrac{1}{2}x.$$

Similarly, with $z = \operatorname{cosec} x$, and as a *corrected* integral,

$$\int^{\frac{1}{2}\pi} \operatorname{cosec} x\,dx = \int_1 \frac{dz}{\surd(z^2-1)} = \cosh^{-1} z$$
$$= \cosh^{-1}\operatorname{cosec} x = \sinh^{-1}\cot x = \tanh^{-1}\cos x.$$

Again, with $$z = \operatorname{sech} x, \text{ or } \operatorname{cosech} x,$$

$$\int_0 \operatorname{sech} x\,dx = \int^1 \frac{dz}{\surd(1-z^2)} = \cos^{-1} z$$
$$= \cos^{-1}\operatorname{sech} x = \sin^{-1}\tanh x = \tan^{-1}\sinh x = 2\tan^{-1}\tanh \tfrac{1}{2}x.$$

$$\int^{\infty} \operatorname{cosech} x\,dx = \int_0 \frac{dz}{\surd(1+z^2)} = \sinh^{-1} z$$
$$= \sinh^{-1}\operatorname{cosech} x = \cosh^{-1}\coth x = \log\coth \tfrac{1}{2}x.$$

199. To integrate $1/P$, where $P = a + b\cos x + c\sin x$, we can proceed in a similar manner, and put $1/P = y$;

and now $$\frac{1}{y} - a = b\cos x + c\sin x,$$

so that $$\frac{1}{y^2}\frac{dy}{dx} = b\sin x - c\cos x = \sqrt{\left\{b^2 + c^2 - \left(\frac{1}{y} - a\right)^2\right\}};$$

and $$\int \frac{dx}{a + b\cos x + c\sin x} = \int \frac{dy}{\surd\{(-a^2+b^2+c^2)y^2 + 2ay - 1\}},$$

which by the results of § 191 can be expressed by

(i.) $$\frac{1}{\surd(a^2-b^2-c^2)} \cos^{-1}\frac{a + (-a^2+b^2+c^2)y}{\surd(b^2+c^2)}$$
$$= \frac{1}{\surd(a^2-b^2-c^2)} \cos^{-1}\frac{aP - a^2 + b^2 + c^2}{\surd(b^2+c^2)P};$$

or (ii.) $$\frac{1}{\surd(-a^2+b^2+c^2)}\cosh^{-1}\frac{aP-a^2+b^2+c^2}{\surd(b^2+c^2)P};$$
as in ex. 34, p. 81.

Similarly to integrate $1/Q$, where
$$Q=a+b\cosh x+c\sinh x,$$
we put $1/Q=y$; and obtain the results of ex. 35, p. 81.

Between the limits 0 and π, (i.) $=\pi/\surd(a^2-b^2-c^2)$ but (ii.) is illusory, as the function to be integrated becomes infinite between the limits of integration.

Again, between the limits 0 and ∞,
$$\int dx/Q \text{ or } \int dx/(a+b\cosh x+c\sinh x)$$
is given by means of ex. 35, p. 81, as
$$\frac{2}{\surd(-a^2+b^2-c^2)}\tan^{-1}\frac{\surd(-a^2+b^2-c^2)}{a+b+c},$$
or $$\frac{2}{\surd(a^2-b^2+c^2)}\begin{matrix}\tanh^{-1}\\ \coth^{-1}\end{matrix}\left\{\frac{\surd(a^2-b^2+c^2)}{a+b+c}\right\}.$$

200. The integration of $(a+b\cos\theta)^{-n}$ is effected

(i.) when $b/a<1$, by the substitution
$$a+b\cos\theta=\frac{a^2-b^2}{a-b\cos v}, \text{ or } \tan\tfrac{1}{2}\theta=\sqrt{\left(\frac{a+b}{a-b}\right)}\tan\tfrac{1}{2}v,$$
equivalent geometrically to a change from the true anomaly θ to the excentric anomaly v (§ 177) in an ellipse of excentricity b/a; and now
$$\int\frac{d\theta}{(a+b\cos\theta)^n}=\frac{1}{(a^2-b^2)^{n-\frac{1}{2}}}\int(a-b\cos v)^{n-1}dv;$$
(ii.) when $b/a>1$, by the substitution
$$a+b\cos\theta=\frac{a^2-b^2}{a-b\cosh u}, \text{ or } \tan\tfrac{1}{2}\theta=\sqrt{\left(\frac{b+a}{b-a}\right)}\tanh\tfrac{1}{2}u,$$
a change from the true anomaly θ to the hyperbolic excentric anomaly u in a hyperbolic orbit of excentricity b/a; and now

$$\int\frac{d\theta}{(a+b\cos\theta)^n}=\frac{1}{(b^2-a^2)^{n-\frac{1}{2}}}\int(b\cosh u-a)^{n-1}du.$$

When $n=1$, we obtain, as in ex. 31, p. 80,

$$\int\frac{d\theta}{a+b\cos\theta}=\frac{v}{\sqrt{(a^2-b^2)}}=\frac{1}{\sqrt{(a^2-b^2)}}\cos^{-1}\frac{a\cos\theta+b}{a+b\cos\theta},$$

or
$$=\frac{u}{\sqrt{(b^2-a^2)}}=\frac{1}{\sqrt{(b^2-a^2)}}\cosh^{-1}\frac{a\cos\theta+b}{a+b\cos\theta}.$$

The integral for $n=2$ is required, as in § 180, for the mean anomaly nt in an elliptic or hyperbolic orbit.

Reciprocally, since $\cos\theta=(a\cosh u-b)/(a-b\cosh u)$,

$$\int\frac{du}{(b\cosh u-a)^n}=\frac{1}{(b^2-a^2)^{n-\frac{1}{2}}}\int(a+b\cos\theta)^{n-1}d\theta,$$

$$\int\frac{du}{(b-a\cosh u)^n}=\frac{1}{(b^2-a^2)^{n-\frac{1}{2}}}\int(a+b\cos\theta)^{n-1}(\sec\theta)^n d\theta;$$

including all possible cases required in the integration of

$$(a+\beta\cosh u)^{-n}.$$

The results for $n=1$ will be found in ex. 32, p. 80.

To integrate $(a+b\sinh u)^{-n}$, substitute

$$a+b\sinh u=(a^2+b^2)/(a-b\sinh v);$$

and now

$$\int\frac{du}{(a+b\sinh u)^n}=\frac{1}{(a^2+b^2)^{n-\frac{1}{2}}}\int(a-b\sinh v)^{n-1}dv.$$

If we put $\theta=\mathrm{gd}u$, $\phi=\mathrm{gd}v$, and $b/a=\tan\gamma$,

then
$$\sinh u=\tan\theta=\frac{a\tan\phi+b}{a-b\tan\phi}=\tan(\phi+\gamma);$$

so that $\theta=\phi+\gamma$; and now

$$\int\frac{du}{(a+b\sinh u)^n}=\frac{1}{(a^2+b^2)^{\frac{1}{2}n}}\int(\cos\theta)^{n-1}(\sec\phi)^n d\theta\ (\text{or } d\phi)$$

and the result for $n=1$ is given in ex. 33, p. 81.

With the notation of the confocal conics of § 177,

$$\tan \tfrac{1}{2}\theta = \coth \tfrac{1}{2}u \tan \tfrac{1}{2}v,\ \tan \tfrac{1}{2}\theta' = \tanh \tfrac{1}{2}u \tan \tfrac{1}{2}v;$$

and taking the logarithmic differentials

$$\frac{d\theta}{\sin\theta} = \frac{-du}{\sinh u} = \frac{dv}{\sin v},\quad \frac{d\theta'}{\sin\theta'} = \frac{du}{\sinh u} = \frac{dv}{\sin v};$$

while

$$(\cosh u - \cos v)(\cosh u + \cos\theta)$$
$$= (\cosh u + \cos v)(\cosh u - \cos\theta') = \sinh^2 u,$$
$$(\cosh u - \cos v)(\cos v - \cos\theta)$$
$$= (\cosh u + \cos v)(\cos\theta' - \cos v) = \sin^2 v;$$

so that

$$\int_0 \frac{(\sinh u)^{2n-1}dv}{(\cosh u + \cos v)^n} = \int_0 (\cosh u - \cos\theta')^n d\theta',$$

$$\int_0 \frac{(\sin v)^{2n-1}du}{(\cosh u + \cos v)^n} = \int_0 (\cos\theta' - \cos v)^n d\theta';\ \text{etc.}$$

As applications the student may evaluate the expression for the area $PpP'p'$ bounded by the elliptic arcs u, u' and the hyperbolic arcs v, v' of the confocals of § 177, or bounded by the circular arcs u, u' and v, v' of the system of dipolar circles of § 175, given by the integral

$$\iint \frac{\partial(x, y)}{\partial(u, v)} du dv = \tfrac{1}{2}c^2 \iint (\cosh 2u - \cos 2v) du dv,$$

or $\displaystyle\iint \frac{c^2 du dv}{(\cosh v + \cos u)^2}$; and determine the centroid.

Prove also that the C.G. is at N (fig. 54, i.) of

(i.) a circular wire in which the line density varies as SP^{-2}; (ii.) a circular area in which the surface density varies as SP^{-4}; (iii.) a spherical shell in which the surface density varies as SP^{-3};

(iv.) a solid sphere in which the volume density varies as SP^{-5}; the boundary in each case being defined by $u =$ a positive constant, so as to include N in the boundary.

Determine also the C.G. of a quadrant of an elliptic wire, as in fig. 55, in which the line density varies as p, $1/p, xp, x/p, \ldots, p$ denoting the length of the perpendicular from the centre O on the tangent.

201. The integrals

$$\text{(i.)} \int (H\cos\theta + K)d\theta/(A\cos^2\theta + 2B\cos\theta + C),$$
$$\text{(ii.)} \int (H\cosh u + K)du/(A\cosh^2 u + 2B\cosh u + C),$$
$$\text{(iii.)} \int (H\sinh u + K)dv/(A\sinh^2 u + 2B\sinh u + C),$$

are reduced to the form of the integral of § 194, by the substitution $x = \cos\theta$, $\cosh u$ or $\sinh v$; and now R is replaced by $\sin\theta$, $\sinh u$, or $\cosh v$; while by writing

$$\sin^2\theta = \{(1-\cos\alpha\cos\theta)^2 - (\cos\theta - \cos\alpha)^2\}/\sin^2\alpha,$$
$$V = A\cos^2\theta + 2B\cos\theta + C$$
$$= P(1-\cos\alpha\cos\theta)^2 + Q(\cos\theta-\cos\alpha)^2,$$

and $$y = \sin^2\alpha\sin^2\theta/V;$$
$$1 - Py = (P+Q)/(\cos\theta - \cos\alpha)^2/V,$$
$$1 + Qy = (P+Q)/(1-\cos\alpha\cos\theta)^2/V;$$

then $1/P$ and $-1/Q$ are the maximum and minimum values of y; and the integration is expressed by inverse sines, circular and hyperbolic, of $\surd(Py)$ and $\surd(Qy)$.

Similarly for $\cosh u$ and $\sinh u$, by writing

$$\sinh^2 u = \{(\cosh\beta\cosh u - 1)^2 - (\cosh u - \cosh\beta)^2\}/\sinh^2\beta,$$
$$\cosh^2 u = \{(\sinh\gamma\sinh u + 1)^2 + (\sinh u - \sinh\gamma)^2\}/\cosh^2\gamma.$$

As numerical examples, integrate

$$\{L(3-\cos\theta) + M(1-3\cos\theta)\}/(5\cos^2\theta - 6\cos\theta + 5)$$
$$\{L(\cosh u - 2) + M(2\cosh u - 1)\}/(5\cosh^2 u - 8\cosh u + 5),$$
$$(H\sinh v + K)/(\sinh^2 v - \sinh v + 1),$$
$$(H\sinh v + K)/(6\sinh^2 v - 4\sinh v + 9).$$

Examples.—Integrate with respect to x,

(1) $\tan x\sec^2 x$, $\tan^2 x\sec^2 x$, $\tan^3 x\sec^2 x$, $(\tan x)^m\sec^2 x$, $\cot x\operatorname{cosec}^2 x$, $\cot^2 x\operatorname{cosec}^2 x$, $(\cot x)^m\operatorname{cosec}^2 x$, $\tan x$, $\cot x$, $\tan^2 x$, $\cot^2 x$, $\tan^3 x$, $\cot^3 x$, $\tan^4 x$, $\cot^4 x$, $\sec x\tan x$, $\sec^2 x\tan x$, $(\sec x)^m\tan x$, $(\operatorname{cosec} x)^m\cot x$, $\sec^2 x$, $\sec^4 x$, $\sec^6 x$, $\operatorname{cosec}^6 x$, $\sec x$, $\sec^3 x$, $\operatorname{cosec}^3 x$, $\sec x\operatorname{cosec} x$, $\operatorname{vers} x$, $\operatorname{vers}^2 x$, $1/\operatorname{vers} x$; also the corresponding hyperbolic functions of x.

(2) $\dfrac{1}{\cos x+\sin x}$, $\dfrac{1}{b\cos x+c\sin x}$, $\dfrac{1}{a+b\tan x}$, $\dfrac{1}{a+b\cos x}$,

$\dfrac{1}{5+4\cos x}$, $\dfrac{1}{4+5\cos x}$, $\dfrac{1}{a^2\cos^2 x\pm\beta^2\sin^2 x}$, $\dfrac{1}{a^2\cosh^2 x\pm\beta^2\sinh^2 x}$,

$\dfrac{B\cos x+C\sin x}{b\cos x+c\sin x}$, $\dfrac{A+B\cos x+C\sin x}{a+b\cos x+c\sin x}$,

$\dfrac{1}{\sin^2 a-\sin^2 x}$, $\dfrac{1}{1-\sin^2 a\sin^2 x}$, $\sec x\sec 2x$, $\tan x\tan 2x$.

(3) Prove that

(i.) $\sin x\sec(x-a)\sec(x-b)$

$=\cos a\cos b\operatorname{cosec}(a-b)\{\sec a\sec(x-a)-\sec b\sec(x-b)\}$;

(ii.) $\dfrac{\sin x}{\sin(x-a)\sin(x-b)\sin(x-c)}=\Sigma\dfrac{\sin a\cot(x-a)}{\sin(a-b)\sin(a-c)}$;

(iii.) $\dfrac{\sin^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)}=\Sigma\dfrac{\sin^2 a\operatorname{cosec}(x-a)}{\sin(a-b)\sin(a-c)}$;

(iv.) $\dfrac{\sin mx}{\sin nx}=\dfrac{1}{2n}\Sigma(-1)^k\sin m\alpha\cot\frac{1}{2}(x-\alpha)$ (where $\alpha=k\pi/n$, and $m<n$)

$=\dfrac{1}{2n}\Sigma(-1)^k\sin m\alpha\operatorname{cosec}(x-\alpha)$, or $\dfrac{1}{2n}\Sigma(-1)^k\sin m\alpha\cot(x-\alpha)$,

according as $m+n$ is odd or even.

Write down the integrals of these functions.

(4) Integrate $\sec(x+a)\sqrt{\{\sec(x+b)\sec(x+c)\}}$, by means of ex. 16, p. 78, putting $\tan x=y$.

(5) Prove that the definite integrals

$$\text{(i.)}\ \int_0^{\frac{1}{4}\pi}\tan x\,dx=\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\cot x\,dx=\log\surd 2=0\cdot 34657.$$

$$\text{(ii.)}\ \int_0^{\frac{1}{4}\pi}\sec x\,dx=\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\operatorname{cosec} x\,dx=\log(\surd 2+1)=0\cdot 88137.$$

$$\text{(iii.)}\ \int_0^{\pi}\frac{dx}{a+b\cos x}=\frac{\pi}{\surd(a^2-b^2)},\ \text{if } a>b.$$

$$\text{(iv.)}\ \int_0^{\frac{1}{2}\pi}\frac{dx}{a+b\cos x}=\frac{1}{\surd(a^2-b^2)}\cos^{-1}\frac{b}{a},$$

$$\text{or}\qquad \frac{1}{\surd(b^2-a^2)}\cosh^{-1}\frac{b}{a}.$$

$$\text{(v.)}\ \int_0^{\frac{1}{2}\pi}\frac{d\theta}{5+3\cos\theta}=0\cdot 23180.$$

$$\text{(vi.)}\ \int_0^{\frac{1}{2}\pi}\frac{d\theta}{3+5\cos\theta}=0\cdot 27465.$$

$$\text{(vii.)}\ \int_0^{\frac{1}{2}\pi}\frac{d\theta}{(5+3\cos\theta)^2}=0\cdot 03494.$$

$$\text{(viii.)}\ \int_0^{\frac{1}{2}\pi}\frac{d\theta}{(3+5\cos\theta)^2}=0\cdot 05267.$$

$$\text{(6) (i.)}\ \int_0^{\pi}\frac{H\cos\theta+K}{\cos^2\theta-2\sin\alpha\cos\theta+1}d\theta=\frac{H\sin\frac{1}{2}\alpha+K\cos\frac{1}{2}\alpha}{\surd(2\cos^3\alpha)}\pi,$$

if $\cos\alpha$ is taken positive.

$$\text{(ii.)}\ \int_0^{\pi}\frac{H\cos\theta+K}{a^2\cos^2\theta-2ab\cos\alpha\cos\theta+b^2}d\theta$$

$$=\frac{H\surd(a^2+b^2-c^2)+K\surd(a^2+b^2+c^2)}{c^2\surd(-a^2+b^2+c^2)}\pi,$$

where $\qquad c^4=a^4-2a^2b^2\cos 2\alpha+b^4.$

202. *Integration by Parts.*

Corresponding to the formula for the differentiation of a product, by parts (§ 12),

$$\frac{duv}{dx}=\frac{du}{dx}v+u\frac{dv}{dx},$$

is the formula, obtained by integrating both sides,

$$uv=\int\frac{du}{dx}vdx+\int u\frac{dv}{dx}dx,$$

or $$\int u\frac{dv}{dx}dx=uv-\int\frac{du}{dx}vdx,$$

or, as it may be written,

$$\int udv=uv-\int vdu,$$

the formula of the Integral Calculus, for *integration by parts*, as it is called.

Interpreted geometrically the formula merely asserts that, as in fig. 19, with u and v for x and y, the rectangle $OMPN$ is the sum of the areas OMP, ONP into which the rectangle is divided by any curved line; or that the area ONP, represented by $\int udv$, is the difference between the rectangle uv, and the area OMP, represented by $\int vdu$.

The formula shows how the integral $\int udv$ may be made to depend upon another integral $\int vdu$, which is either integrable, or which is made to depend upon another integral, by another integration by parts.

Thus (i.) to integrate xe^x, take $x=u$, $\dfrac{dv}{dx}=e^x$; then $v=e^x$,

and $$\int xe^xdx=xe^x-\int e^xdx=xe^x-e^x.$$

(ii.) to integrate $\log x$, take $u=\log x$, $dv=dx$, $v=x$; then $\int\log xdx=x\log x-\int xdx/x=x\log x-x$.

(iii.) $\sin\sqrt{x}$, put $\sqrt{x}=z$, then

$$\int\sin\sqrt{x}dx=\int\sin z\,.\,2zdz$$

$$= -2z\cos z + 2\int \cos z dz = -2z\cos z + 2\sin z$$
$$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x}.$$

(iv.) $$\int \sin^{-1}x dx = x\sin^{-1}x - \int\frac{x dx}{\sqrt{(1-x^2)}}$$
$$= x\sin^{-1}x + \sqrt{(1-x^2)}.$$

(v.) to integrate $\cos mx\cosh nx$; here $\cos mx$ or $\cosh nx$ may be taken indiscriminately for u, and we must integrate by parts twice.

For instance if $\cos mx = u$, $\cosh nx = \frac{dv}{dx}$, $v = \frac{\sinh nx}{n}$,

$$\int \cos mx\cosh nx dx = \frac{1}{n}\cos mx\sinh nx + \frac{m}{n}\int \sin mx\sinh nx dx$$

$$= \frac{1}{n}\cos mx\sinh nx + \frac{m}{n^2}\sin mx\cosh nx - \frac{m^2}{n^2}\int \cos mx\cosh nx dx;$$

and therefore, by transposition and division,

$$\int \cos mx\cosh nx dx = \frac{m\sin mx\cosh nx + n\cos mx\sinh nx}{m^2+n^2} \quad \text{(i.)}$$

In a similar manner

$$\int \cos mx\sinh nx dx = \frac{m\sin mx\sinh nx + n\cos mx\cosh nx}{m^2+n^2} \quad \text{(ii.)}$$

$$\int \sin mx\cosh nx dx = \frac{n\sin mx\sinh nx - m\cos mx\cosh nx}{m^2+n^2} \quad \text{(iii.)}$$

$$\int \sin mx\sinh nx dx = \frac{n\sin mx\cosh nx - m\cos mx\sinh nx}{m^2+n^2} \quad \text{(iv.)}$$

By addition of (i.) and (ii.), (iii.) and (iv.), or independently by integration by parts,

$$\int e^{nx}\cos mx dx = e^{nx}\frac{n\cos mx + m\sin mx}{m^2+n^2} \text{..................(v.)}$$

$$\int e^{nx}\sin mx dx = e^{nx}\frac{n\sin mx - m\cos mx}{m^2+n^2} \text{..................(vi.)}$$

With a subsidiary angle α given by $\tan\alpha = m/n$,

$$\int e^{nx}\cos mx\,dx = e^{nx}\cos(mx-\alpha)\cos\alpha/n,$$

$$\int e^{nx}\sin mx\,dx = e^{nx}\sin(mx-\alpha)\cos\alpha/n.$$

Again, by successive integration by parts,

$$\int e^{x/a}u\,dx = ae^{x/a}\left(u - a\frac{du}{dx} + a^2\frac{d^2u}{dx^2} - a^3\frac{d^3u}{dx^3} + \dots\right).$$

$$\int u\frac{d^nv}{dx^n}dx = u\frac{d^{n-1}v}{dx^{n-1}} - \frac{du}{dx}\frac{d^{n-2}v}{dx^{n-2}} + \dots$$

$$+(-1)^{n-1}\frac{d^{n-1}u}{dx^{n-1}}v + (-1)^n\int\frac{d^nu}{dx^n}v\,dx;$$

the integrated terms of which can again be integrated by parts.

We can establish Taylor's Theorem by successive integration by parts; for

$$\mathrm{f}(a+h) - \mathrm{f}a = \int_a^{a+h}\mathrm{f}'x\,dx;$$

$$= -\{(a+h-x)\mathrm{f}'x\}_a^{a+h} + \int_a^{a+h}(a+h-x)\mathrm{f}''x\,dx$$

$$= h\mathrm{f}'a - \left\{\frac{1}{2!}(a+h-x)^2\mathrm{f}''x\right\}_a^{a+h} + \frac{1}{2!}\int_a^{a+h}(a+h-x)^2\mathrm{f}'''x\,dx$$

$$= h\mathrm{f}'a + \frac{h^2}{2!}\mathrm{f}''a + \dots + \frac{h^n}{n!}\mathrm{f}^na \quad + \frac{1}{n!}\int_a^{a+h}(a+h-x)^n\mathrm{f}^{n+1}x\,dx;$$

so that, as in § 114,

$$\mathrm{f}(a+h) - \mathrm{f}a - h\mathrm{f}'a - \frac{h^2}{2!}\mathrm{f}''a - \dots - \frac{h^n}{n!}\mathrm{f}^na$$

$$= R = \frac{1}{n!}\int_a^{a+h}(a+h-x)^n\mathrm{f}^{n+1}x\,dx.$$

It is assumed here that $\mathrm{f}x$, $\mathrm{f}'x$, $\mathrm{f}''x, \dots \mathrm{f}^n x$, $\mathrm{f}^{n+1}x$ are all finite between the limits a and $a+h$ of x; and now if P denotes a certain average value of $\mathrm{f}^{n+1}x$ between the limits a and $a+h$, which we may denote by $\mathrm{f}^{n+1}(a+\theta h)$,

$$R=\frac{1}{n!}\int_a^{a+h}(a+h-x)^n P dx=\frac{h^{n+1}}{(n+1)!}\mathrm{f}^{n+1}(a+\theta h).$$

Examples.—Integrate by parts,

x^2e^x, x^3e^{ax}, $x^4e^{ax+b}, \dots$;
$e^{\sqrt{x}}$, $e^{\sqrt[3]{x}}$, $\exp\sqrt[n]{x}$;
$x^m\log x^n$, $(\log x)^2$, $(\log x)^3$, $(\log x)^4$, $(\log x)^5, \dots$;
$x^2\cos x$, $x^3\sin x, \dots$; $\cos\sqrt[3]{x}$, $\sin\sqrt[4]{x}, \dots$;
$e^{ax+b}\cos(mx+n)$, $e^{ax+b}\sin(mx+n), \dots$;
$\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$, $\operatorname{vers}^{-1}x$;
$\cosh^{-1}x$, $\sinh^{-1}x$, $\tanh^{-1}x, \dots$;
$(\sin^{-1}x)^2$, $(\sinh^{-1}x)^2$, $(\cosh^{-1}x)^2$, $(\operatorname{vers}^{-1}x)^2$;
$\tan^{-1}\sqrt{x}$, $\tanh^{-1}\sqrt[3]{x}, \dots$;
$\sqrt{(a^2-x^2)}$, $\sqrt{(x^2-a^2)}$, $\sqrt{(a^2+x^2)}$, $(a^2-x^2)^{\frac{3}{2}}$,
$x^2\sqrt{(a^2-x^2)}, \dots$;
$x\sin^{-1}x$, $x^2\cos^{-1}x$, $x^3\cosh^{-1}x \dots$;
$\log x \,.\, \sin^{-1}x$, $\log x \,.\, \cosh^{-1}x$;
$(\sec x)^3$, $(\sec x)^5$, $(\operatorname{cosec} x)^5, \dots \sec x\tan^2x$, $\sec x\tan^4x, \dots$

203. The integration of $\sqrt{(a^2-x^2)}$, $\sqrt{(x^2-a^2)}$, $\sqrt{(a^2+x^2)}$ is required in the quadrature of the circle, ellipse, and hyperbola (§§ 50-54); thus for example, integrating by parts for the quadrature of the circle,

$$\int\sqrt{(a^2-x^2)}dx=x\sqrt{(a^2-x^2)}+\int\frac{x^2dx}{\sqrt{(a^2-x^2)}}$$

$$=x\sqrt{(a^2-x^2)}+a^2\int\frac{dx}{\sqrt{(a^2-x^2)}}-\int\sqrt{(a^2-x^2)}dx,$$

(and, by transposition and division)

$$=\tfrac{1}{2}\sqrt{(a^2-x^2)}+\tfrac{1}{2}a^2\sin^{-1}(x/a) \text{ (§ 50).}$$

Similarly $\int\sqrt{}(\alpha - x\,.\,x - \beta)dx$

$$=\tfrac{1}{2}\left(x-\frac{\alpha+\beta}{2}\right)\sqrt{}(\alpha-x\,.\,x-\beta)+\tfrac{1}{4}(\alpha-\beta)^2\tan^{-1}\frac{x-\beta}{\alpha-x};$$

so that taken between the limits α and β, at which $\sqrt{}(\alpha-x\,.\,x-\beta)$ vanishes, and between which it is real, the integral is $\frac{1}{8}\pi(\alpha-\beta)^2$.

More generally, denoting $ax^2+2bx+c$ by R, and integrating by parts

$$\int\sqrt{}Rdx=\tfrac{1}{2}\frac{ax+b}{a}\sqrt{}R-\tfrac{1}{2}\frac{b^2-ac}{a}\int\frac{dx}{\sqrt{}R};$$

and the result has one of the three forms of ex. 28, p. 80.

When a is negative and b^2-ac positive, the value of x is restricted to lie between the roots, α and β suppose, of the equation $R=0$, for R to be positive and $\sqrt{}R$ real; and now, taken between the limits α and β,

$$\int_\alpha^\beta\sqrt{}Rdx=\frac{ac-b^2}{\sqrt{}(-a^3)}\frac{\pi}{2}.$$

Suppose for instance that the area of the ellipse is required, when it is given by the general equation of the second degree (§ 13)

$$ax^2+2\,hxy+by^2+2\,gx+2\,fy+c=0.$$

Solving this equation as a quadratic in y,

$$hx+by+f=\sqrt{}\{(h^2-ab)x^2+2(fh-bg)x+f^2-bc\};$$

so that (§ 49)

$$b(y_1-y_2)=2\sqrt{}\{(h^2-ab)x^2+2(fh-bg)x+f^2-bc\};$$

and the area of the ellipse is $\int(y_1-y_2)dx$, taken between the limits for which the quantity under the radical is positive; and therefore the area is $\pi\Delta/(ab-h^2)^{\frac{3}{2}}$, where Δ denotes the *discriminant* (ex. 9, p. 139).

This result must be multiplied by sin ω, if the coordinate axes are inclined at an angle ω.

204. *Formulas of Reduction.*

A *formula of reduction* in the Integral Calculus is a formula, obtained in general by integration by parts, by which one integral, say u_n, is made to depend upon a simpler integral, say u_{n-1}, or u_{n-2}; then by successive substitution in the formula of reduction, we finally arrive at an integration which can be effected.

Suppose, for instance, $u_n = \int(\sin\theta)^n d\theta$; integrating by parts, with $u=(\sin\theta)^{n-1}$, $dv=\sin\theta d\theta$, $v=-\cos\theta$,

$$\begin{aligned} u_n &= \int(\sin\theta)^n d\theta = \int(\sin\theta)^{n-1}\sin\theta d\theta \\ &= -(\sin\theta)^{n-1}\cos\theta + (n-1)\int(\sin\theta)^{n-2}(\cos\theta)^2 d\theta \\ &= \dots\dots\dots\dots\dots\dots + (n-1)u_{n-2} - (n-1)u_n; \end{aligned}$$

or

$$nu_n = \dots\dots\dots\dots\dots\dots + (n-1)u_{n-2},$$

$$u_n = -\frac{1}{n}(\sin\theta)^{n-1}\cos\theta + \frac{n-1}{n}u_{n-2},$$

a *formula of reduction.*

Similarly

$$\int(\cos\theta)^n d\theta = \frac{1}{n}\sin\theta(\cos\theta)^{n-1} + \frac{n-1}{n}\int(\cos\theta)^{n-2}d\theta,$$

another formula of reduction.

Taken between the limits 0 and $\frac{1}{2}\pi$,

$$\int_0^{\frac{1}{2}\pi}(\sin\theta)^n d\theta = \frac{n-1}{n}\int_0^{\frac{1}{2}\pi}(\sin\theta)^{n-2}d\theta,$$

$$\int_0^{\frac{1}{2}\pi}(\cos\theta)^n d\theta = \frac{n-1}{n}\int_0^{\frac{1}{2}\pi}(\cos\theta)^{n-2}d\theta.$$

First suppose n is an even integer $2m$; then

$$\begin{aligned} \int_0^{\frac{1}{2}\pi}\left(\overset{\sin}{\underset{\cos}{\theta}}\right)^{2m} d\theta &= \frac{2m-1}{2m}\int_0^{\frac{1}{2}\pi}\left(\overset{\sin}{\underset{\cos}{\theta}}\right)^{2m-2} d\theta \\ &= \frac{2m-3}{2m-2}\,\frac{2m-1}{2m}\int_0^{\frac{1}{2}\pi}\left(\overset{\sin}{\underset{\cos}{\theta}}\right)^{2m-4} d\theta, \end{aligned}$$

and so on, finally being

$$\int_0^{\frac{1}{2}\pi}\left({}^{\sin}_{\cos}\,\theta\right)^{2m}d\theta=\frac{1\,.\,3\,.\,5\,\ldots\,2m-1}{2\,.\,4\,.\,6\,\ldots\quad 2m}\,\frac{\pi}{2}.$$

Secondly, suppose n an odd integer $2m+1$; then

$$\int_0^{\frac{1}{2}\pi}\left({}^{\sin}_{\cos}\,\theta\right)^{2m+1}d\theta=\frac{2\,.\,4\,.\,6\,\ldots\quad 2m}{3\,.\,5\,.\,7\,\ldots\,2m+1}\int_0^{\frac{1}{2}\pi}{}^{\sin}_{\cos}\,\theta d\theta.$$

$$=\frac{2\,.\,4\,.\,6\,\ldots\quad 2m}{3\,.\,5\,.\,7\,\ldots\,2m+1}.$$

These are called *Wallis's Theorems*, and are of great practical use in the Integral Calculus.

Both cases are included in the formula

$$\int_0^{\frac{1}{2}\pi}\left({}^{\sin}_{\cos}\,\theta\right)^{n}d\theta=\frac{(n-1)(n-3)\ldots}{n(n-2)(n-4)\ldots}(\tfrac{1}{2}\pi)^{\frac{1}{2}+\frac{1}{2}(-1)^n}$$

If we put $\cos\theta=\operatorname{sech}u$, or $\theta=\operatorname{gd}u$, then $d\theta=\operatorname{sech}u\,du$;

and $\displaystyle\int_0^{\frac{1}{2}\pi}(\cos\theta)^n d\theta=\int_0^{\infty}(\operatorname{sech}u)^{n+1}du$; so that

$$\int_0^{\infty}(\operatorname{sech}u)^{2m+1}du=\frac{1\,.\,3\,.\,5\,\ldots\,2m-1}{2\,.\,4\,.\,6\,\ldots\quad 2m}\,\frac{\pi}{2};$$

$$\int_0^{\infty}(\operatorname{sech}u)^{2m}du\;=\frac{2\,.\,4\,.\,6\,\ldots\,2m-2}{3\,.\,5\,.\,7\,\ldots\,2m-1}.$$

When m or n is not an integer, these integrals depend upon the Gamma Function, defined in the next article.

Examples.—Prove the following formulas of reduction, and evaluate the integrals for $n=1, 2, 3, 4, \ldots$.

(1) $u_n=\int x^n e^x dx=x^n e^x-n\int x^{n-1}e^x dx=x^n e^x-nu_{n-1}$.

(2) $u_n=\int(\log x)^n dx=x(\log x)^n-nu_{n-1}$.

(3) $u_n=\displaystyle\int x^m(\log x)^n dx=\frac{x^{m+1}(\log x)^n}{m+1}-\frac{nu_{n-1}}{m+1}$.

(4) $u_n=\int x^n\sqrt{(a^2-x^2)}dx=-\frac{x^{n-1}(a^2-x^2)^{\frac{3}{2}}}{n+2}+\frac{n-1}{n+2}a^2u_{n-2}$.

(5) $u_n=\int x^n\sqrt{(2ax-x^2)}dx$

$$=-\frac{x^{n-1}(2ax-x^2)^{\frac{3}{2}}}{n+2}+\frac{2n+1}{n+2}au_{n-1}.$$

(6) $u_n=\int(a^2-x^2)^{n-\frac{1}{2}}=\{x(a^2-x^2)^{n-\frac{1}{2}}+(2n-1)a^2u_{n-1}\}/2n$.

(7) $u_n=\int\frac{dx}{(Ax^2+2Bx+C)^n}$

$$=\frac{1}{2n-2}\,\frac{1}{AC-B^2}\,\frac{Ax+B}{(Ax^2+2Bx+C)^{n-1}}+\frac{2n-3}{2n-2}\,\frac{Au_{n-1}}{AC-B^2}.$$

(8) $u_n=\int\frac{dx}{(x^p+a^v)^n}=\frac{x}{(n-1)pa^p(x^p+a^p)^{n-1}}+\frac{np-p-1}{(n-1)p}\,\frac{u_{n-1}}{a^p}$.

(9) $u_n=\int\frac{dx}{(x-a)^n\sqrt{(x-b)}}$

$$=\frac{1}{n-1}\,\frac{1}{b-a}\,\frac{\sqrt{(x-b)}}{(x-a)^{n-1}}+\frac{2n-3}{2n-2}\,\frac{u_{n-1}}{b-c}.$$

(10) $u_n=\int x^n dx/\sqrt{R}$

$$=\{x^{n-1}\sqrt{R}-(2n-1)bu_{n-1}-(n-1)cu_{n-2}\}/na.$$

(11) Determine L, M, N in the formulas of reduction

$$u_n=\int\frac{dx}{x^n\sqrt{R}}=\frac{L\sqrt{R}}{x^{n-1}}+Mu_{n-1}+Nu_{n-2}.$$

(12) $u_n=\int\frac{dx}{(x-q)^n\sqrt{R}}=\frac{L\sqrt{R}}{(x-q)^{n-1}}+Mu_{n-1}+Nu_{n-2}$.

(13) $u_n=\int(\tan x)^n dx=\frac{(\tan x)^{n-1}}{n-1}-u_{n-2}$.

(14) $u_n=\int(\cot x)^n dx=-\frac{(\cot x)^{n-1}}{n-1}-u_{n-2}$.

(15) $u_n=\int(\sec x)^n dx=\frac{(\sec x)^{n-2}\tan x}{n-1}+\frac{n-2}{n-1}u_{n-2}$.

(16) $u_n=\int(\operatorname{cosec} x)^n dx=-\frac{(\operatorname{cosec} x)^{n-2}\cot x}{n-1}+\frac{n-2}{n-1}u_{n-2}$.

(17) $$u_n=\int(\operatorname{vers} x)^n dx=-\frac{\sin x(\operatorname{vers} x)^{n-1}}{n}+\frac{2n-1}{n}u_{n-1}.$$

(18) $$u_n=\int x^n\cos mx=\frac{x^n}{m}\sin mx+\frac{nx^{n-1}}{m^2}\cos mx-\frac{n(n-1)}{m^2}u_{n-2}.$$

(19) $$u_n=\int e^{px}(\cos x)^n dx=\frac{e^{px}(\cos x)^{n-1}(p\cos x+n\sin x)}{n^2+p^2}$$
$$+\frac{n(n-1)}{n^2+p^2}u_{n-2}.$$

(20) $$v_{(m,\,n)}=\int(\sin x)^m(\cos x)^n dx$$
$$=\frac{(\sin x)^{m+1}(\cos x)^{n-1}}{m+n}+\frac{n-1}{m+n}u_{(m,\,n-2)},$$

or $$-\frac{(\sin x)^{m-1}(\cos x)^{n+1}}{m+n}+\frac{m-1}{m+n}u_{(m-2,\,n)},$$

or $$\frac{(m-1)(\sin x)^{m+1}(\cos x)^{n-1}-(n-1)(\sin x)^{m-1}(\cos x)^{n+1}}{(m+n)(m+n-2)}$$
$$+\frac{(m-1)(n-1)u_{(m-2,\,n-2)}}{(m+n)(m+n-2)},$$

(21) $$\int_0^{\frac{1}{2}\pi}(\sin x)^m(\cos x)^{2n+1}dx=\frac{2\,.\,4\,.\,6\quad\dots\quad 2n}{(m+1)(m+3)\dots(m+2n+1)}.$$

(22) $$\int_0^{\frac{1}{2}\pi}(\sin x)^{2m}(\cos x)^{2n}dx=\frac{1.3.5\dots(2m-1).1.3.5\dots(2n-1)}{2.4.6\dots2m(2m+2)\dots(2m+2n)}\frac{\pi}{2}.$$

(23) $$u_n=\int\frac{d\theta}{(a+b\cos\theta)^n}=-\frac{b\sin\theta}{(n-1)(a^2-b^2)(a+b\cos\theta)^{n-1}}$$
$$+\frac{(2n-3)au_{n-1}-(n-2)u_{n-}}{(n-1)(a^2-b^2)}.$$

Investigate formulas of reduction for the integration of $(\sin^{-1}x)^n$, $(\operatorname{vers}^{-1}x)^n$, $(\cosh x)^n$, $(\sinh x)^n$, $(\tanh x)^n$, $(\operatorname{sech} x)^n$, $(\sinh^{-1}x)^n$, $(\cosh^{-1}x)^n$, $\sin\sqrt[n]{x}$, $\sinh\sqrt[n]{x}$, $(a+b\cosh u)^{-n}$, $(a+b\cos\theta+c\sin\theta)^{-n}$, $(a+b\cosh u+c\sinh u)^{-n}$.

205. *Definite Integrals.*

Any integral taken between definite fixed lin called a *definite* integral (§ 45), but the name is used especially for a large class of integrals which can only be evaluated between certain limits, and of which the *indefinite* integrals (§ 39) cannot be found.

The subject of Definite Integrals is a large one, and would lead us beyond the scope of the present treatise; we may however consider a few elementary definite integrals, which are of frequent occurrence, and great practical use.

Consider first the definite integral

$$\int_0^\infty e^{-x}x^{n-1}dx, \text{ or } \int_0^1 (-\log z)^{n-1}dz = \int_0^1 (\log 1/z)^{n-1}dz,$$

if $e^{-x}=z$; this is called the Gamma function, because it is usually denoted by Γn.

By integration by parts

$$\Gamma n = \left(-e^{-x}x^{n-1}\right)_0^\infty + (n-1)\int_0^\infty e^{-x}x^{n-2}dx = (n-1)\Gamma(n-1);$$

so that if n is a positive integer,

$$\Gamma n = (n-1)!\,\Gamma 1 = (n-1)!$$

since $$\Gamma 1 = \int_0^\infty e^{-x}dx = 1, \text{ also } \Gamma 0 = \infty.$$

Changing x into x^2 gives

$$\Gamma m = 2\int_0^\infty e^{-x^2}\,x^{2m-1}dx;$$

so that we may express the product of Γm and Γn by

$$\Gamma m\Gamma n = 4\int_0^\infty e^{-x^2}x^{2m-1}dx\int_0^\infty e^{-y^2}y^{2n-1}dy$$

$$=4\int_0^\infty\int_0^\infty e^{-x^2-y^2}x^{2m-1}y^{2n-1}dxdy$$

$$=4\int_0^\infty e^{-r^2}r^{2m+2n-1}dr\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta$$

(on changing to polars, with $x=r\cos\theta$, $y=r\sin\theta$)

$$=2\Gamma(m+n)\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta.$$

The definite integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx=2\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta,$$

on putting $x=(\sin\theta)^2$; and is called the *First Eulerian Integral*, and denoted by $B\ (m,\ n)$; so that

$$\Gamma m\Gamma n=\Gamma(m+n)B(m,\ n).$$

Also, on substituting

$$\alpha-x=(\alpha-\beta)\sin^2\theta,\ x-\beta=(\alpha-\beta)\cos^2\theta,$$

$$\int_\beta^\alpha(\alpha-x)^{m-1}(x-\beta)^{n-1}dx=(\alpha-\beta)^{m+n-1}B(m,\ n).$$

This integral has been evaluated for positive integral values of m and n (p. 424); and for fractional values it depends on the tabulated values of the Gamma Function, by means of the above relation

$$B(m,\ n)=\Gamma m\Gamma n/\Gamma(m+n).$$

When $m+n=1$, then (§ 189)

$$\Gamma m\,\Gamma(1-m)=B(m,\ 1-m)=2\int_0^{\frac{1}{2}\pi}(\tan\theta)^{2m-1}d\theta=\pi\operatorname{cosec}m\pi.$$

Putting $m=\frac{1}{2}$, gives $\Gamma\frac{1}{2}=\sqrt{\pi}$, and $\Gamma\frac{3}{2}=\frac{1}{2}\Gamma\frac{1}{2}=\frac{1}{2}\sqrt{\pi}$; and now if Γm is calculated from $m=0{\cdot}5$ to $m=1$, then Γm is known from $m=0{\cdot}5$ to $m=0$; and thence Γn or $\log\Gamma n$ can be tabulated from $n=1$ to $n=2$ (Bertrand, *Calcul Intégral*, p. 285), and the graph of Γx can be drawn for all positive values of x.

The substitution of $x=z/(1+z)$ or $1/(1+z)$ shows that

$$B(m,\ n)=\int_0^\infty \frac{z^{m-1}dz}{(1+z)^{m+n}}=\int_0^\infty \frac{z^{n-1}dz}{(1+z)^{m+n}}=B(n,m).$$

As an application, let us determine the area A of the positive quadrant of the curve $x^n+y^n=a^n$; then

$$A=\int_0^a (a^n-x^n)^{\frac{1}{n}}dx=\frac{2}{n}a^2\int_0^{\frac{1}{2}\pi}(\sin\theta)^{\frac{2}{n}-1}(\cos\theta)^{\frac{2}{n}+1}d\theta$$

(on substituting $\qquad x^n=a^n\sin^2\theta$)

$$=\frac{a^2}{n}B\left(\frac{1}{n},\frac{1}{n}+1\right)=\frac{a^2}{n}\Gamma\frac{1}{n}\Gamma\left(\frac{1}{n}+1\right)\Big/\Gamma\left(\frac{2}{n}+1\right)$$

$$=\frac{a^2}{2n}\left(\Gamma\frac{1}{n}\right)^2\Big/\Gamma\frac{2}{n}.$$

For instance, if $n=2$, we obtain the area of the quadrant of the circle $\frac{1}{4}\pi a^2$.

Also $\qquad \bar{x}=\bar{y}=\frac{2}{3}a\left(\Gamma\frac{2}{n}\right)^2\Big/\left(\Gamma\frac{1}{n}\Gamma\frac{3}{n}\right).$

By integrating, as in § 134, throughout the space in which $x,\ y,\ z$ are positive, but

$$(x/a)^m+(y/b)^n+(z/c)^p-1$$

is negative, we can show in a similar manner, that the volume V of this space within the surface is given by

$$\frac{V}{abc}=\Gamma\left(\frac{1}{m}+1\right)\Gamma\left(\frac{1}{n}+1\right)\Gamma\left(\frac{1}{p}+1\right)\Big/\Gamma\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{p}+1\right)$$

while

$$\frac{\bar{x}V}{a^2bc}=\frac{1}{2}\Gamma\left(\frac{2}{m}+1\right)\Gamma\left(\frac{1}{n}+1\right)\Gamma\left(\frac{1}{p}+1\right)\Big/\Gamma\left(\frac{2}{m}+\frac{1}{n}+\frac{1}{p}+1\right),\ldots$$

agreeing with § 134 when we put $m=n=p=2$.

Again, the length of a branch of the curve $r^n=a^n\cos n\theta$ (§ 172) can be expressed by $\frac{a}{n}B\left(\frac{1}{2n},\ \frac{1}{2}\right)$.

206. *Bernoulli's and Euler's Numbers as Definite Integrals.*

Making use of the relation, with $mu=x$,

$$\int_0^\infty e^{-mu}u^{p-1}du=m^{-p}\int_0^\infty e^{-x}x^{p-1}dx=m^{-p}\Gamma p,$$

then, with the notation of § 112,

$$T_p\Gamma p=\int_0^\infty (e^{-u}+e^{-3u}+e^{-5u}+\ldots)u^{p-1}du$$

$$=\int_0^\infty \frac{e^{-u}u^{p-1}du}{1-e^{-2u}}=\frac{1}{2}\int_0^\infty \operatorname{cosech} u\,.\,u^{p-1}du\,;$$

so that $$S_p=\frac{2^pT_p}{2^p-1}=\frac{2^{p-1}}{(2^p-1)\Gamma p}\int_0^\infty \operatorname{cosech} u\,.\,u^{p-1}du\,;$$

and $$B_n=\frac{2n}{(2^{2n}-1)\pi^{2n}}\int_0^\infty \operatorname{cosech} u\,.\,u^{2n-1}du\,;$$

so that, on putting $u=\frac{1}{2}\pi v$,

$$\int_0^\infty \operatorname{cosech} \tfrac{1}{2}\pi v\,.\,v^{2n-1}dv=2^{2n}(2^{2n}-1)B_n/2n,$$

the coefficient of $x^{2n-1}/(2n-1)!$ in the expansion of $\tan x$, and this is always an integer; namely 1, 2, 16, 272, ...

Similarly (§§ 112, 113)

$$U_p\Gamma p=\int_0^\infty (e^{-u}-e^{-3u}+e^{-5u}\ldots)u^{p-1}du=\frac{1}{2}\int \operatorname{sech} u\,.\,u^{p-1}du\,;$$

$$E_n=2(2n)!U_{2n+1}/(\tfrac{1}{2}\pi)^{2n+1}=\int_0^\infty \operatorname{sech} \tfrac{1}{2}\pi v\,.\,v^{2n}dv\,;$$

which, since E_n is an integer, shows that this definite integral is always an integer, namely 1, 5, 61, 1385, ...

Put $\operatorname{sech}\frac{1}{2}\pi v=\cos\phi$, $\frac{1}{2}\pi v=\log(\sec\phi+\tan\phi)$,

and we find (§ 113)

$$\int_0^{\frac{1}{2}\pi}\{\log(\sec\phi+\tan\phi)\}^{2n}d\phi=(\tfrac{1}{2}\pi)^{2n+1}E_n=2(2n)!\,U_{2n+1},$$

$$\int_0^{\frac{1}{2}\pi}\{\log(\sec\phi+\tan\phi)\}^{2n-1}d\phi=2(2n-1)!\,U_{2n}.$$

Thus, for example, the area between a branch of the curve $m\theta=\operatorname{gd}(r/a)$, or $r/a=\log(\sec m\theta+\tan m\theta)$, and its asymptote is

$$\tfrac{1}{2}a^2\int_0^{\frac{1}{2}\pi/m}\{\log(\sec m\theta+\tan m\theta)\}^2d\theta$$

$$=\tfrac{1}{2}a^2(\tfrac{1}{2}\pi)^3E_1/m=\tfrac{1}{16}\pi^3a^2/m.$$

207. *Differentiation and Integration of a Definite Integral, and Applications.*

Denoting the indefinite integral with respect to x of a function $\mathrm{f}(x, c)$, involving a parameter c (§ 105), by $\mathrm{f}_1(x, c)$, then the definite integral

$$I=\int_b^a\mathrm{f}(x, c)dx=\mathrm{f}_1(a, c)-\mathrm{f}_1(b, c)\,;$$

and now, if a, b, c are functions of some variable t,

$$\frac{dI}{dt}=\frac{\partial I}{\partial a}\frac{da}{dt}+\frac{\partial I}{\partial b}\frac{db}{dt}+\frac{\partial I}{\partial c}\frac{dc}{dt}$$

$$=\mathrm{f}(a, c)\frac{da}{dt}-\mathrm{f}(b, c)\frac{db}{dt}+\int_b^a\frac{\partial\mathrm{f}(x, c)}{\partial c}dx\frac{dc}{dt}\,;$$

of which a geometrical interpretation can easily be constructed.

We can also integrate I according to a similar rule.

By means of this method of differentiation and integration we can show the connexion between different classes of integrals, as already done in §§ 193, 194, and also verify the solution of certain differential equations, when the solution is given as a definite integral, as in § 184.

Thus, if α and β are positive, and $I = \frac{1}{\surd\alpha\beta}\tan^{-1}\sqrt{\frac{\beta}{\alpha}}$,

$$\int_0^{\frac14\pi}\frac{d\theta}{\alpha\cos^2\theta+\beta\sin^2\theta} = \frac{1}{2\surd(\alpha\beta)}\cos^{-1}\left(\frac{\alpha\cos^2\theta-\beta\sin^2\theta}{\alpha\cos^2\theta+\beta\sin^2\theta}\right)_0^{\frac14\pi} = I,$$

$$\int_0^{\infty}\frac{du}{\alpha\cosh^2u+\beta\sinh^2u} = \frac{1}{2\surd(\alpha\beta)}\cos^{-1}\left(\frac{\alpha\cosh^2u-\beta\sinh^2u}{\alpha\cosh^2u+\beta\sinh^2u}\right)_0^{\infty} = I;$$

and
$$\int_0^{\frac14\pi}\frac{d\theta}{(\alpha\cos^2\theta+\beta\sin^2\theta)^{n+1}} = \frac{(-1)^n}{n!}\left(\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\right)^2 I,$$

$$\int_0^{\infty}\frac{du}{(\alpha\cosh^2u+\beta\sinh^2u)^{n+1}} = \frac{(-1)^n}{n!}\left(\frac{\partial}{\partial\alpha}-\frac{\partial}{\partial\beta}\right)^n I.$$

We can verify that the definite integral

$$\int_0^{\pi}\cos(pv - x\sin v)dv,$$

which we have denoted by $\pi J_p(x)$ in § 184, and called the *Bessel function* of x of the pth order, is a solution of Bessel's differential equation given on p. 185.

For dividing the definite integral into two parts

$$y = \int_0^{\pi}\cos pv\cos(x\sin v)dv, \text{ and } z = \int_0^{\pi}\sin pv\sin(x\sin v)dv;$$

and using accents to denote differentiation,

$$\begin{aligned} & x^2y'' + xy' + (x^2-p^2)y \\ &= \int_0^{\pi}\{-x^2\sin^2v\cos pv\cos(x\sin v) - x\sin v\cos pv\sin(x\sin v) \\ &\qquad\qquad + (x^2-p^2)\cos pv\cos(x\sin v)\}dv \\ &= \{x\cos v\cos pv\sin(x\sin v) - p\sin pv\cos(x\sin v)\}_0^{\pi} \\ &= p\sin p\pi; \end{aligned}$$

while
$$\begin{aligned} & x^2z'' + xz' + (x^2-p^2)z \\ &= \{-x\cos v\sin pv\cos(x\sin v) + p\cos pv\sin(x\sin v)\}_0^{\pi} \\ &= -x\sin p\pi; \end{aligned}$$

and therefore y and z each satisfy Bessel's equation if p is an integer; and y or z vanishes as p is odd or even.

The student may verify in a similar manner that if $P_n(\mu)$ (p. 314) satisfies the differential equation of ex. 6, p. 276,

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP}{d\mu}\right\}+n(n+1)P=0,$$

$$\pi P_n(\mu)=\int_0^\pi\{\mu-\surd(\mu^2-1)\cos\phi\}^n d\phi$$

$$=\int_0^\pi\{\mu+\surd(\mu^2-1)\cos\psi\}^{-n-1}d\psi,$$

the one integral being transformed into the other by (§ 200)

$$\mu-\surd(\mu^2-1)\cos\phi=\frac{1}{\mu+\surd(\mu^2-1)\cos\psi}=\frac{d\phi}{d\psi}.$$

Putting $\mu=\cos u$, or $\cosh v$, according as μ is < or > 1,

$$\frac{d^2P}{du^2}+\cot u\frac{dP}{du}+n(n+1)P=0,$$

or
$$\frac{d^2P}{dv^2}+\coth v\frac{dP}{dv}-n(n+1)P=0;$$

so that the solution may be written

$$\pi P_n(\cos u)=\int_0^\pi(\cos u-i\sin u\cos\phi)^n d\phi,$$

or
$$\pi P_n(\cosh v)=\int_0^\pi(\cosh v-\sinh v\cos\phi)^n d\phi.$$

Another solution Q_n is obtained by putting

$$Q_n(\mu)=\int_0^\infty\{\mu+\surd(\mu^2-1)\cosh\phi\}^{-n-1}d\phi,$$

or
$$\int_0^{\coth^{-1}\mu}\{\mu-\surd(\mu^2-1)\cosh\psi\}^n d\psi,$$

by means of the substitution (§ 200)

$$\{\mu+\surd(\mu^2-1)\cosh\phi\}\ \{\mu-\surd(\mu^2-1)\cosh\psi\}=1;$$

and
$$Q_n(\mu)=\infty \text{ when } \mu=0.$$

Example.—Prove

$$\frac{1}{\cosh v-\cos u}=\sum^\infty(2n+1)P_n(\cos u)Q_n(\cosh v).$$

208. Mr. W. M. Hicks has shown (*Phil. Trans.*, 1881) that similar functions are required in the solution of $\nabla^2 V=0$ (§ 146), in connexion with the anchor rings or *toroidal* surfaces, generated by the revolution of the dipolar circles of § 175.

With the columnar coordinates y, θ, x (§ 132) round Ox as axis, we obtain the transformation

$$-\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial V}{\partial y}\right)+\frac{1}{y^2}\frac{\partial^2 V}{\partial \theta^2};$$

or, on putting $V=\psi y^{-\frac{1}{2}}$,

$$-\nabla^2 V=y^{-\frac{1}{2}}\left\{\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}+\frac{1}{y^2}\left(\frac{\partial^2 \psi}{\partial \theta^2}+\tfrac{1}{4}\psi\right)\right\};$$

and, on changing from x, y to new variables u, v, conjugate functions given by $u+iv=\mathrm{f}(x+iy)$ (§ 133)

$$-\nabla^2 V=Jy^{-\frac{1}{2}}\left\{\frac{\partial^2 \psi}{\partial u^2}+\frac{\partial^2 \psi}{\partial v^2}+\frac{1}{Jy^2}\left(\frac{\partial^2 \psi}{\partial \theta^2}+\tfrac{1}{4}\psi\right)\right\}.$$

With the dipolars of § 175,

$$x+iy=c\tan\tfrac{1}{2}(u+iv),$$

$$\frac{1}{J}=\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2=\frac{c^2}{(\cos u+\cosh v)^2}=y^2\operatorname{cosech}^2 v,$$

$$-\nabla^2 V=y^{-\frac{5}{2}}\left\{\left(\frac{\partial^2 \psi}{\partial u^2}+\frac{\partial^2 \psi}{\partial v^2}\right)\sinh^2 v+\frac{\partial^2 \psi}{\partial \theta^2}+\tfrac{1}{4}\psi\right\}.$$

Then, if ψ varies as $\cos nu \cos m\theta$,

$$-\nabla^2 V=y^{-\frac{5}{2}}\sinh^2 v\left\{\frac{\partial^2 \psi}{\partial v^2}-n^2\psi-(m^2-\tfrac{1}{4})\psi\operatorname{cosech}^2 v\right\}.$$

Further, if we put $\psi=\sqrt{(c\sinh v)}P$, $-\nabla^2 V=$

$$\frac{(\cos u+\cosh v)^{\frac{5}{2}}}{c^2}\left(\frac{\partial^2 P}{\partial u^2}+\frac{\partial^2 P}{\partial v^2}+\coth v\frac{\partial P}{\partial v}+\operatorname{cosech}^2 v\frac{\partial^2 P}{\partial \theta^2}+\tfrac{1}{4}P\right).$$

If we suppose $P=P_n^m \cos nu \cos m\theta$ is a solution of $\nabla^2 V=0$, then P_n^m, as a function of v only, satisfies the differential equation

$$\frac{d^2P}{dv^2}+\coth v\frac{dP}{dv}-(n^2-\tfrac{1}{4})P-m^2P\operatorname{cosech}^2 v=0;$$

and this equation, in the particular case when $m=0$, is the same as that satisfied by the *zonal harmonic* (p. 276) with $n-\frac{1}{2}$ for n; so that writing C for $\cosh v$ or μ, and S for $\sinh v$ or $\surd(\mu^2-1)$, the solution is AP_n+BQ_n,

where $\pi P_n=\int_0^\pi \frac{d\phi}{(C+S\cos\phi)^{n+\frac{1}{2}}}$, $\pi Q_n=\int_0^\infty \frac{d\phi}{(C+S\cosh\phi)^{n+\frac{1}{2}}}$.

For, as in ex. 6, v., p. 276,

$$\begin{aligned} S^2\frac{dP_n}{dC} &= -\int_0^\pi \frac{(n+\frac{1}{2})(S^2+SC\cos\phi)d\phi}{(C+S\cos\phi)^{n+\frac{3}{2}}} \\ &=(n+\tfrac{1}{2})\int \frac{1-C(C+S\cos\phi)}{(C+S\cos\phi)^{n+\frac{3}{2}}}d\phi \\ &=(n+\tfrac{1}{2})(P_{n+1}-CP_n); \end{aligned}$$

while taking the form $\pi P_n=\int_0^\pi (C-S\cos\phi)^{n-\frac{1}{2}}d\phi$,

$$S^2\frac{dP_n}{dC}=(n-\tfrac{1}{2})(CP_n-P_{n-1});$$

and thence $$\frac{d}{dC}\left(S^2\frac{dP_n}{dC}\right)=(n^2-\tfrac{1}{4})P_n;$$

and similarly with Q_n.

209. In the general case, when m is any integer, we can verify that the solution of

$$\frac{d}{dC}\left(S^2\frac{dP}{dC}\right)-(n^2-\tfrac{1}{4})P-\frac{m^2P}{S^2}=0$$

is $$P=AP_n^m+BQ_n^m,$$

where

$$\pi P_n^m = \int_0^\pi \frac{\cos m\phi\, d\phi}{(C+S\cos\phi)^{n+\frac{1}{2}}}, \quad \pi Q_n^m = \int_0^\infty \frac{\cosh m\phi\, d\phi}{(C+S\cosh\phi)^{n+\frac{1}{2}}}.$$

For now, taking Q_n^m, we prove that

$$\left(S^2\frac{d}{dC}-mC\right)Q_n^m+(n-m-\tfrac{1}{2})SQ_n^{m+1}$$

$$=-\left\{\frac{S\sinh(m+1)\phi+C\sinh m\phi}{\pi(C+S\cosh\phi)^{n+\frac{1}{2}}}\right\}_0^\infty=0, \text{ if } m<n;$$

and thus, as in § 87,

$$S^m\frac{d^mQ_n}{dC^m}=S\frac{d}{dC}\left(S\frac{d}{dC}-\frac{C}{S}\right)\left(S\frac{d}{dC}-2\frac{C}{S}\right)\dots\left(S\frac{d}{dC}-\overline{m-1}\,.\,\frac{C}{S}\right)Q_n$$

$$=(-1)^m(n-\tfrac{1}{2})(n-\tfrac{3}{2})\dots(n-m-\tfrac{1}{2})Q_n^m.$$

Similarly

$$\left(S^2\frac{d}{dC}+mC\right)Q_n^m+(n+m-\tfrac{1}{2})SQ_n^{m-1}$$

$$=\left\{\frac{S\sinh(m-1)\phi+C\sinh m\phi}{\pi(C+S\cosh\phi)^{n+\frac{1}{2}}}\right\}_0^\infty=0;$$

and thence $\quad \dfrac{d}{dC}\left(S^2\dfrac{d}{dC}Q_n^m\right)$

$$=-mQ_n^m-mC\frac{d}{dC}Q_n^m-(n+m-\tfrac{1}{2})\left(\frac{C}{S}Q_n^{m-1}+S\frac{d}{dC}Q_n^{m-1}\right)$$

$$=\quad\left(n^2-\tfrac{1}{4}+\frac{m^2}{S^2}\right)Q_n^m,$$

thus verifying the differential equation; and the procedure is the same with P_n^m.

The substitution $\tan\frac{1}{2}\theta=\dfrac{\sqrt{(C+S\cosh\phi)}}{\cosh\frac{1}{2}u\sinh\phi}$ will give

$$\int_0^\pi\frac{d\theta}{(C-\cos\theta)^{\frac{1}{2}}}=\int_0^\infty\frac{\sqrt{2}d\phi}{(C+S\cosh\phi)^{\frac{1}{2}}}=\sqrt{2}\pi Q_0;$$

and generally if we denote $\displaystyle\int_0^\pi\frac{\cos n\theta d\theta}{(C-\cos\theta)^{\frac{1}{2}}}$ by πR_n, then

$$S^2\frac{dR_n}{dC}-(n+\tfrac{1}{2})(R_{n+1}-CR_n)=\left\{\frac{C\sin n\theta-\sin(n+1)\theta}{(C-\cos\theta)}\right\}_0^\pi=0,$$

$$S^2\frac{dR_n}{dC}-(n-\tfrac{1}{2})(CR_n-R_{n-1})=\left\{\frac{-C\sin n\theta+\sin(n-1)\theta}{(C-\cos\theta)^{\frac{1}{2}}}\right\}_0^\pi=0,$$

n being an integer; so that R_n satisfies the same equations as Q_n; and since $R_0=\surd 2Q_0$, therefore $R_n=\surd 2Q_n$.

By differentiation with respect to C,

$$\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\ldots(m-\tfrac{1}{2})\int_0^\pi\frac{S^m\cos n\theta d\theta}{(C-\cos\theta)^{m+\frac{1}{2}}}$$

$$=\pi(-1)^mS^m\frac{d^mR_n}{dC^m}=\pi\surd 2(-1)^mS^m\frac{d^mQ_n}{dC^m}$$

$$=\pi\surd 2(n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots(n-m-\tfrac{1}{2})Q_n^m$$

$$=\surd 2(n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots(n-m-\tfrac{1}{2})\int_0^\infty\frac{\cosh m\phi d\phi}{(C+S\cosh\phi)^{n+\frac{1}{2}}}.$$

When the dipolar system of § 175 is revolved round Oy, we shall find in the same manner, on putting

$$V=\psi x^{-\frac{1}{2}},\text{ and }\psi=\surd(c\sin u)P,$$

$$-\nabla^2V=x^{-\frac{5}{2}}\left\{\left(\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}\right)\sin^2u+\frac{\partial^2\psi}{\partial\theta^2}+\tfrac{1}{4}\psi\right\}$$

$$=\frac{(\cos u+\cosh v)^{\frac{5}{2}}}{c^2}\left(\frac{\partial^2P}{\partial u^2}+\cot u\frac{\partial P}{\partial u}+\frac{\partial^2P}{\partial v^2}+\operatorname{cosec}^2u\frac{\partial^2P}{\partial\theta^2}-\tfrac{1}{4}P\right);$$

and now if $P=P_n^m\cos nv\cos m\theta$ is a solution of $\nabla^2V=0$, then P_n^m, as a function of u only, satisfies the equation

$$\frac{d^2P}{du^2}+\cot u\frac{dP}{du}-(n^2+\tfrac{1}{4})P-m^2P\operatorname{cosec}^2u=0,$$

of which the solution is obtained from the preceding results by replacing $\cosh v$ and $\sinh v$ by $\cos u$ and $\sin u$.

In the particular case of $m=0$, the order of the corresponding zonal harmonics would be $-\tfrac{1}{2}+in$, and therefore imaginary; these are called *Mehler functions.*

210. *Lagrange's Theorem.*

We can establish Lagrange's Theorem (§ 150) by the rule for the differentiation of a definite integral; for, if

$$y = a + x\phi y,$$

$$\frac{1}{n!}\frac{\partial}{\partial a}\int_a^y (x\phi z + a - z)^n \mathrm{f}'z\,dz$$

$$= \frac{1}{(n-1)!}\int_a^y (x\phi z + a - z)^{n-1}\mathrm{f}'z - \frac{x^n}{n!}(\phi a)^n \mathrm{f}'a;$$

so that differentiating again $n-1$ times with respect to a, and denoting

$$\frac{1}{n!}\frac{\partial^n}{\partial a^n}\int_a^y (x\phi z + a - z)^n \mathrm{f}'z\,dz \text{ by } R_n,$$

we find that

$$R_n = R_{n-1} - \frac{x^n}{n!}\frac{\partial^{n-1}}{\partial a^{n-1}}\left\{(\phi a)^n \mathrm{f}'a\right\},$$

while $R_0 = \mathrm{f}y - \mathrm{f}a$; so that

$$R_n = \mathrm{f}y - \mathrm{f}a - \sum_{r=1}^{r=n}\frac{x^r}{r!}\frac{\partial^{r-1}}{\partial a^{r-1}}\left\{(\phi a)^r \mathrm{f}'a\right\},$$

which is Lagrange's Theorem.

211. *Integration considered as Summation.*

Hitherto we have considered Integration as an operation to be performed by the reversion of Differentiation; but now, by the aid of Taylor's Theorem, we can exhibit it in its fundamental conception as the Summation of a number of infinitesimal elements, and show that the result is the same as before.

For let any function $\mathrm{f}x$, between the fixed values a and b of x, be considered for the consecutive infinitesimally graduated sequence of values $a, x_1, x_2, \dots x_r, \dots x_n, b$, of the variable x.

Then by Taylor's Theorem (§§ 114, 118), using the symbol Δx_r to denote the difference $x_{r+1}-x_r$,

$$\mathrm{f}x_1-\mathrm{f}a-\Delta a\,\mathrm{f}'a=\tfrac{1}{2}\Delta a^2\,\mathrm{f}''(a+\theta\,\Delta a),$$
$$\mathrm{f}x_2-\mathrm{f}x_1-\Delta x_1\mathrm{f}'x_1=\tfrac{1}{2}\Delta x_1{}^2\mathrm{f}''(x_1+\theta_1\Delta x_1),$$

....

$$\mathrm{f}x_{r+1}-\mathrm{f}x_r-\Delta x_r\mathrm{f}'x_r=\tfrac{1}{2}\Delta x_r{}^2\mathrm{f}''(x_r+\theta_r\Delta x_r),$$

....

$$\mathrm{f}b-\mathrm{f}x_n-\Delta x_n\mathrm{f}'x_n=\tfrac{1}{2}\Delta x_n{}^2\mathrm{f}''(x_n+\theta_n\Delta x_n).$$

Adding these equations, and using the symbol Σ to denote summation for all integral values of r from 1 to n,

$$\mathrm{f}b-\mathrm{f}a-\Sigma\Delta x_r\mathrm{f}'x_r=\Sigma\tfrac{1}{2}\Delta x_2{}^2\mathrm{f}''(x_r+\theta_r\Delta x_r).$$

Now if o denotes the greatest of all the evanescent quantities, such as $\Delta x_r\mathrm{f}''(x_r+\theta_r\Delta x_r)$, then

$\mathrm{f}b-\mathrm{f}a-\Sigma\mathrm{f}'x_r\Delta x_r$ is less than $\tfrac{1}{2}o\Sigma(x_{r+1}-x_r)$ or $\tfrac{1}{2}o(b-a)$; and this quantity ultimately vanishes; so that replacing Σ and Δ by $\int$ and d, as in § 44,

$$\mathrm{f}b-\mathrm{f}a-\int_a^b\mathrm{f}'x\,dx=0,$$

which establishes the fact that Integration considered as Summation is a process the reverse of Differentiation.

For instance in Quadrature (§ 41), we determine the arithmetic mean of the ordinates of a curve, and call this the mean ordinate; and now the area between any two ordinates is a rectangle of the same breadth, and height equal to the mean ordinate.

Suppose however that M, the geometric mean of the equispaced ordinates of the curve $y=\mathrm{f}x$ between $x=a$ and $x=b$ is required; then $M=\mathrm{lt}\sqrt[n]{(\mathrm{f}x_1\,.\,\mathrm{f}x_2\ldots\mathrm{f}x_r\ldots\mathrm{f}x_n)}$,

and $$\log M=\mathrm{lt}\,\frac{1}{n}\Sigma\log\mathrm{f}x=\mathrm{lt}\,\Sigma\log\mathrm{f}x\Delta x/(b-a)$$
$$=\int_a^b\log\mathrm{f}x\,dx/(b-a),$$

so that $$M=\exp\left\{\int_a^b \log \mathrm{f}x\,dx/(b-a)\right\}.$$

As an application, prove that the geometric mean of the ordinates of the curve $y=\sin x$, from 0 to π is $\frac{1}{2}$.

212. *Approximate Quadrature.*

Various rules are employed in practice for determining A the area of an irregular figure, when a Planimeter is not used.

The simplest is that usually employed with an Indicator Diagram; the breadth of the closed contour is measured at a number of equidistant ordinates which divide the area into a number n of strips, and the arithmetic mean of these is taken as the mean breadth; more strictly we take

$$A=l(\tfrac{1}{2}y_0+y_1+y_2+\ldots+y_{n-1}+\tfrac{1}{2}y_n),$$

where the y's denote the breadth of the figure at the different ordinates, and l denotes the spacing of the ordinates; this is equivalent to replacing the area by trapezoids, bounded by the same ordinates.

In Simpson's rule a higher approximation is obtained by taking the curve through the tops of three equidistant ordinates as a parabola with axis parallel to the ordinates; and now if y_{m-1}, y_m, y_{m+1} denote three consecutive ordinates and l their spacing, the parabola is taken as given by $$y=a+bx+\tfrac{1}{2}cx^2;$$
and if the origin is taken at the foot of the middle ordinate, as in § 77,

$$a=y_m,\ 2bl=y_{m+1}-y_{m-1},\ cl^2=y_{m+1}-2y_m+y_{m-1}.$$

The area of the two strips is now given by

$$\int_{-l}^{l} y\,dx=2al+\tfrac{1}{3}cl^3=\tfrac{1}{3}l(y_{m+1}+4y_m+y_{m-1});$$

and $$A=\tfrac{1}{3}l(y_0+4y_1+2y_2+4y_3+\ldots+4y_{2n}+y_{2n+1}),$$
which is called *Simpson's Rule.*

We may replace the boundary by the curve of the third degree $y = a + bx + \frac{1}{2}cx^2 + \frac{1}{6}dx^3$, passing through the tops of four equidistant ordinates $y_m, y_{m+1}, y_{m+2}, y_{m+3}$; and now the area of the three strips

$$\int_0^{3l} y\,dx = 3l(a + \tfrac{3}{2}bl + \tfrac{3}{2}cl^2 + \tfrac{9}{8}el^3)$$

$$= \tfrac{3}{8}l(y_m + 3y_{m+1} + 3y_{m+2} + y_{m+3}).$$

It is convenient to take an odd number of ordinates and an even number of strips, and a general formula for approximate quadrature, with $2n+1$ ordinates and $2n$ strips, has been given by Newton.

With 7 ordinates and 6 strips, the expression for the area is, very approximately,

$$A = \tfrac{3}{10}l\{6u_m + u_{m-1} + u_{m+1} + 5(u_{m-2} + u_{m+2}) + u_{m-3} + u_{m+3}\},$$

called *Weddle's Rule.*

In these expressions for the area, the y's may represent breadths as well as ordinates; or they may represent the areas of parallel equidistant sections of a solid body; for instance, horizontal and vertical sections of a ship.

Simpson's Rule may also be used for finding the centroid and moment of inertia of the area, by putting

$$A\bar{x} = \int xy\,dx = \tfrac{1}{3}l(x_0y_0 + 4x_1y_1 + 2x_2y_2 + \dots),$$

$$A\bar{y} = \int \tfrac{1}{2}y^2dx = \tfrac{1}{6}l(y_0^2 + 4y_1^2 + 2y_2^2 + \dots);$$

$$Ak_x^2 = \int \tfrac{1}{3}y^3dx = \tfrac{1}{9}l(y_0^3 + 4y_1^3 + 2y_2^3 + \dots),$$

$$Ak_y^2 = \int x^2y\,dx = \tfrac{1}{3}l(x_0^2y_0 + 4x_1^2y_1 + 2x_2^2y_2 + \dots).$$

As a numerical application, calculate these quantities for the horizontal section of a ship, given the half ordinates from the median line as ·5, 6, 10, 12, 12·4, 12·5, 12·5, 12·5, 12·4, 12·3, 11, 8, and ·5 feet; the ordinates being spaced at 12 feet.

213. *Differential Equations.*

In Chapters III. and V. the principles of Differentiation have been illustrated in the formation of certain important *Differential Equations* from their primitive relations; but now it is requisite to reverse this process, and to discover the primitive relation or solution where possible, when the differential equation is given.

The complete subject of Differential Equations would soon lead us beyond the scope of the present work, and the reader is referred to the treatises of Forsyth and Woolsey Johnson; at the same time, however, it is possible to indicate the procedure in a few simple cases, often required in Dynamics and Electricity.

The simplest case of frequent occurrence is the *linear differential equation with constant coefficients*, of the form $\frac{d^ny}{dx^n}+A_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots+A_{n-1}\frac{dy}{dx}+A_ny=V$,

where the A's are constant, and V is a given function of x.

Using D to denote the operation d/dx, we may write the equation $\mathrm{F}(D)y=V$,..........................(1).
where $\mathrm{F}(D)$ is a rational integral algebraical function of D; and now the solution will consist of two parts, one of them being any *particular solution* of this equation (1), while the other, called the *complementary function*, is the general solution of the equation

$$\mathrm{F}(D)y=0\text{.............................(2).}$$

We have shown in § 68 that $\mathrm{F}(D)e^{bx}=e^{bx}\mathrm{F}b$, so that $y=e^{bx}$ will be a solution of (2) if $\mathrm{F}b=0$; and if the n roots $b_1, b_2, b_3, \ldots$ of this equation $\mathrm{F}b=0$ are all real and distinct, the complementary function will be

$$y=B_1e^{b_1x}+B_2e^{b_2x}+\ldots+B_ne^{b_nx},$$

where $B_1, B_2, \ldots, B_n$ are n arbitrary constants.

If m roots are equal to b, the corresponding terms in the complementary function are, as in ex. 3, p. 143,

$$(C_0 + C_1 x + \dots + C_{m-1} x^{m-1}) e^{bx}.$$

If a pair of roots are conjugate imaginary, $a \pm i\beta$, the corresponding terms, by De Moivre's theorem, are written

$$e^{ax}(A \cos \beta x + B \sin \beta x);$$

and if these imaginary roots are repeated m times, the corresponding terms are

$$e^{ax} \sum_{s=0}^{s=m-1} x^s (A_s \cos \beta x + B_s \sin \beta x).$$

In determining the *particular solution* of (1), we write the equation symbolically

$$y = (\mathrm{F}D)^{-1} V,$$

where $(\mathrm{F}D)^{-1}$ can be resolved into a series of partial fractions of the form $A(D-b)^{-1}$, or $B(D-b)^{-r}$.

Consider the separate terms of V, of the form x^n, e^{ax}, e^{bx}, $\sin mx$, ...

(i.) $(\mathrm{F}D)^{-1} x^n$ is found by expanding $(\mathrm{F}D)^{-1}$ in ascending powers of D as far as D^n, and performing the differentiations; terms beyond D^n may be neglected, since $D^{n+1} x^n = 0$, ...

(ii.) $$(\mathrm{F}D)^{-1} e^{ax} = e^{ax} (\mathrm{F}a)^{-1},$$

and $$(\mathrm{F}D)^{-1} e^{ax} \mathrm{f}x = e^{ax} \{\mathrm{F}(D+a)\}^{-1} \mathrm{f}x, \text{ (§ 70).}$$

(iii.) Thus, if $\mathrm{F}b = 0$,

$$(D-b)^{-r} e^{bx} = e^{bx} D^{-r} . 1 = e^{bx}\left(C_0 + C_1 x + \dots + C_{r-1} x^{n-1} + \frac{x^r}{r!}\right),$$

since D^{-r} represents the operation of integration, repeated r times.

(iv.) $$\begin{aligned}(D-b)^{-r} \cos mx &= (D+b)^r (D^2 - b^2)^{-r} \cos mx \\ &= (D+b)^r (-m^2 - b^2)^{-r} \cos mx,\end{aligned}$$

(§ 68); and so on for $(D^2 + m^2)^{-r} \cos mx, \dots$, by employing the exponential values of $\cos mx$, ... (§ 111).

Consider for instance the differential equation

$$\frac{d^2x}{dt^2}+2n\cos\beta\frac{dx}{dt}+n^2x=E\cos pt,$$

for the *forced vibrations*, due to $E\cos pt$, of a system of which the natural motion is given by the complementary function $x=ae^{-nt\cos\beta}\cos(nt\sin\beta+\epsilon)$ (ex. 6, p. 139), representing S.H.M. (§ 103) of period $2\pi/(n\sin\beta)$, with a *modulus of decay* $n\cos\beta$.

The particular solution is

$$x=\frac{E\cos pt}{D^2+2n\cos\beta\,.\,D+n^2}$$

$$=\frac{D^2-2n\cos\beta\,.\,D+n^2}{(D^2+n^2)^2-4n^2\cos^2\beta\,.\,D^2}E\cos pt$$

$$=\frac{(n^2-p^2)E\cos pt+2np\cos\beta\,.\,E\sin pt}{(n^2-p^2)^2+4n^2p^2\cos^2\beta}$$

$$=\frac{E\cos[pt-\tan^{-1}\{2np\cos\beta/(n^2-p^2)\}]}{\sqrt{(n^4+2n^2p^2\cos 2\beta+p^4)}},$$

representing a S.H.M. of amplitude

$$E/\sqrt{(n^4+2n^2p^2\cos 2\beta+p^4)},$$

and change of *phase* $\tan^{-1}\{2np\cos\beta/(n^2-p^2)\}$.

Examples.—Solve the differential equations

(i.) $y''-5y'+6y=x^2+e^{4x}+\sin 4x+xe^{2x}+x^2e^{3x}$,
(ii.) $y''+4y=\sin x+\sin 2x$,
(iii.) $y''-6y'+9y=\cosh 3x+x\cos 2x$,
(iv.) $y''+2y'+5y=e^x\sin 2x+\cosh x\cos 2x$,
(v.) $y'''+y''-y'-y=xe^x+\cosh x$,
(vi.) $y'''-8y''+16y'=\sinh 2x+\cosh 4x$,
(vii.) $y'''+y'+10y=\cosh 2x+\sinh x\sin 2x$,
(viii.) $y'''-26y'-60y=\sinh 6x+(A+B\cosh 3x)\cos x$,
(ix.) $y''''+2y''+x=\sin 2x+\sin x$,
(x.) $y''''-2y'''+3y''-2y'+y=x+e^{-\frac{1}{2}x}(A+B\cos\frac{1}{2}\sqrt{3}x)$,
(xi.) $y''''+32y'+48y=x\sinh 2x+\cosh 2x\cos(2\sqrt{2}x)$.

214. With variable coefficients P, Q, R, ..., given functions of x, the general linear differential equation of the first order is written $y'+Py=Q$.

To solve this, we can proceed as in § 88, and put $y=uv$, when
$$u'v+u(v'+Pv)=Q\,;$$
and now if v is chosen so that
$$v'+Pv=0, \text{ or } v=\exp(-\textstyle\int Pdx),$$
then
$$u'=Q\exp\textstyle\int Pdx,$$
and
$$u=\textstyle\int Q(\exp\int Pdx)dx+C,$$
$$y=uv=\exp(-\textstyle\int Pdx)\{\int Q(\exp\int Pdx)dx+C\}.$$

Thus, for instance, the solution of the equation
$$L\frac{di}{dt}+Ri=E\cos pt,$$
connecting R the resistance in *ohms*, i the current in *ampères*, $E\cos pt$ the variable electromotive force of a dynamo in *volts*, and Ldi/dt the counter electromotive force of induction, so that L is the inductance in *secohms* (Fleming, *The Alternate Current Transformer*) is
$$i=e^{-Rt/L}(\textstyle\int E\cos pt\,.\,e^{Rt/L}dt+C)$$
$$=\frac{RE\cos pt+pLE\sin pt}{R^2+p^2L^2}+Ce^{-Rt/L}$$
$$=\frac{E\cos\{pt-\tan^{-1}(pL/R)\}}{\surd(R^2+p^2L^2)}+Ce^{-Rt/L}.$$

But considered as a differential equation with constant coefficients, $Ce^{-Rt/L}$ is the complementary function, while the particular solution is, as before,
$$i=\frac{E\cos pt}{LD+R}=\frac{R-LD}{R^2-L^2D^2}E\cos pt$$
$$=\frac{RE\cos pt+pLE\sin pt}{R^2+p^2L^2}=\frac{E\cos\{pt-\tan^{-1}(pL/R)\}}{\surd(R^2+p^2L^2}.$$

215. The general linear differential equation of the second order (§ 88) $y''+2Py'+Qy=R$, on putting $y=uv$, and choosing v so that

$$v'+Pv=0,$$

becomes

$$u''+Iu=S,$$

where

$$I=Q-P^2-P',\ S=R/v;$$

and now, if we can determine the complementary function Au_1+Bu_2, then, as in § 88,

$$u_1'u_2-u_1u_2'=C;$$

and the particular solution is

$$Cu=u_1\int Su_2dx-u_2\int Su_1dx.$$

Thus, as on pp. 183-185, Bessel functions are required in the solution of Riccati's equation, in which $I=kx^m$; and the student can easily verify that

$$u=Ax\int_0^{\frac{1}{2}\pi}{}^{\cos}_{\cosh}(mx^n\sin\theta)(\cos\theta)^{\frac{1}{n}}d\theta$$

$$+B\int_0^{\infty}{}^{\cosh}_{\cos}(mx^n\sinh\phi)(\cosh\phi)^{-\frac{1}{n}}d\phi$$

is the solution when $I=\pm m^2n^2x^{2n-2}$, with certain restrictions on the value of n.

Again, *zonal harmonics* (pp. 276, 314, 431) give the solution when

$$I=(n+\tfrac{1}{2})^2+\tfrac{1}{4}\operatorname{cosec}^2u,\text{ or }-(n+\tfrac{1}{2})^2+\tfrac{1}{4}\operatorname{cosech}^2v;$$

and the *toroidal* and *Mehler functions* of §§ 208, 209, when

$$I=-n^2-(m^2-\tfrac{1}{4})\operatorname{cosech}^2v,\text{ or }-n^2-(m^2-\tfrac{1}{4})\operatorname{cosec}^2u.$$

But no general method has yet been discovered for the solution of the linear differential equation of the second order in the canonical form

$$u''+Iu=0,$$

where I is an arbitrary function of x; nor *a fortiori* for differential equations of a higher order.

216. *Simultaneous differential equations* with constant coefficients, as those given in exs. 3 and 4, p. 219, may be solved by assuming $x = Pe^{bt}$, $y = Qe^{bt}$, or $x = P\sin(nt + \epsilon)$, $y = Q\sin(nt + \epsilon')$, and eliminating P/Q, and ϵ, ϵ', when an equation for the determination of b or n will be obtained.

Thus for instance, in the theory of the Induction Coil, the differential equations

$$L\frac{di_1}{dt} + M\frac{di_2}{dt} + Ri_1 = E\cos pt,$$

$$M\frac{di_1}{dt} + N\frac{di_2}{dt} + Si_2 = 0,$$

for the primary current i_1, and the secondary current i_2, are solved by assuming

$$i_1 = I_1\cos(pt + \epsilon_1),\ i_2 = I_2\cos(pt + \epsilon_2),$$

and determining I_1, I_2, ϵ_1, ϵ_2.

The formation of *partial* differential equations has been illustrated in Chap. V., from which the solution of simple partial differential equations of the first order may be inferred; but for further developments the reader must consult the Treatises on Differential Equations mentioned above.